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ABSTRACT

This teacher's guide provides background information, lists of needed equipment, suggestions for instruction, and answers to student exercises. The unit is designed for elementary school students as a supplement to other SMSG volumes; it can, however, be used in any program. It is designed to make mathematics enjoyable, promote systematic thinking, practice and reinforce arithmetic skills, and provide opportunity for independent research. (RH)

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MATHEMATICS
STUDY GROUP

PROBABILITY FOR INTERMEDIATE GRADES

Teacher's Commentary

(Revised Edition)

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PROBABILITY FOR INTERMEDIATE GRADES

Teacher's Commentary

(Revised Edition)

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PROBABILITY FOR INTERMEDIATE GRADES

Introduction

Why Should We Teach Young Children About Probability?

There are many reasons for helping children develop ideas concerned with probability.

Children, as well as adults, are often confronted with situations which involve probability. While these confrontations may not be crucial to children, they are important and are frequent, so a fundamental knowledge of probability is useful.

There is a need on the part of both children and adults to be better informed in this field. It is evident that both have many misconceptions and a lack of background in probability. Many people have little intuition about experiences involving chance.

Children can learn the fundamentals of probability. Because the approach taken in this unit is of an intuitive nature and not a formal mathematical one, children can understand and profit from the activities and materials that are included.

The practical uses of probability are many, and they are increasing almost daily. Probability is used in making decisions in such diverse fields as military operations, scientific research, design and quality control of manufactured products, insurance calculations, business predictions, weather forecasing, and governmental operations.

Probability is, of course, an essential element in all games of chance. In fact, the theory of probability had its beginning some 300 years ago as a result of a mathematician's interest in the imprudent and unsuccessful gambling activities of one of his friends.

"Goals for School Mathematics", the report of the 1963 Cambridge Conference, recommended that some of the basic ideas of probability be introduced very early in the school program. The School Mathematics Study Group undertook the development of such materials for grades kindergarten to high school.

What Is the Purpose of This Unit?

The ideas of chance and uncertainty are meaningful to children at an early age. They observe and play games with spinners, dice and cards. They hear and use the words, "likely", "unlikely", "chance", and "probably". They recognize that some things are more or less likely to occur than are other things. This recognition is sometimes the result of experience but more often it is intuitive in nature. It is seldom based on any thoughtful or systematic consideration of the situation.

Children's natural interest in games provides a high level of motivation for the study of probability. This unit capitalizes on that interest and motivation by using experimental activities to introduce some of the basic ideas of probability. As children classify and tabulate data gathered from these experimental activities, they are guided to discover patterns which enable them to formulate hypotheses and reach tentative conclusions about situations which involve chance events. Ultimately they are led to discover that there are mathematical justifications for many of their intuitive ideas and that mathematics can be used to test the validity of their intuitive ideas. We hope that this unit will make further study of probability easier, more interesting, and more fruitful.

How Does This Unit Fit Into the Mathematics Program for the Intermediate Grades?

Aside from the fact that the study of probability is an increasingly important part of mathematics, there are certain fringe benefits to be derived from this unit.

First, while the unit has sufficient depth to challenge the able pupil, it should be fun for all children. Making mathematics enjoyable increases pupil motivation and develops a positive attitude toward the subject. The use of experimental activities and games as an integral part of the unit should help those children who heretofore have not enjoyed mathematics to become more interested and to experience unexpected success.

Second, one of the goals of instruction is to promote systematic thinking rather than a hit-or-miss approach which encourages jumping to conclusions. Efforts are made throughout the unit to demonstrate the advantages of a logical and systematic approach.

Third, there are unlimited opportunities for the practice and reinforcement of arithmetic skills ranging from simple addition facts to the multiplication of common fractions. The construction and interpretation of graphs should contribute to increased pupil understanding in areas such as economics, geography, and history, as well as mathematics.

Fourth, there are opportunities for independent investigation -- one might almost say research -- by individual children. The teacher, too, may seize opportunities to carry the work further when individual or group interests indicate that this is desirable. Also, the Appendix in the pupil book which describes probability devices may be of help in such situations.

How Might I Best Use This Unit?

First, you will want to read the next section, "Mathematical Comments". This is a brief introduction to probability and enables you to have a broader view of the topic. Further background can be gained from the two SMSG volumes, Introduction To Probability - Part I - Basic Concepts and Introduction To Probability - Part II - Special Topics. These units which are designed for the junior high school are an excellent source of information. You will find Part I to be especially helpful.

Next, examine the contents of this unit to see the various ideas that are introduced and the suggestions for helping children develop them. Look at the pupil pages to determine the kinds of activities that the children will be doing.

Because it is assumed that pupils have had no previous instruction concerning probability, this unit is not designated for a specific grade. So it will not precisely "fit" your class. With certain lessons and parts of lessons, you will need to adapt, to emphasize more, to emphasize less, to move rapidly, to progress slowly, in short to proceed as you know is best in your situation. The teaching suggestions are labeled, "Suggested Procedure", and are not intended to be prescriptive. They do, however, follow a logical, and experience has shown, a teachable sequence. Generally, you will find a successful approach is to follow the sequence as it is developed but to vary the pace and to omit or to provide more experiences and to stop, as is appropriate for your children's interests and abilities. The unit can be taught in a "concentrated" approach over a short period of time but probably children will learn best if it is fitted in with your present mathematics program over a longer period of time.

What Materials and Equipment Will I Need?

This list indicates the material that is needed. The three types of spinners are available with this unit. You may want to substitute other materials for those suggested here.

	Number Needed
I. BASIC	
A. Spinners	
1. $\frac{1}{2}$ red, $\frac{1}{2}$ blue	6
2. $\frac{1}{4}$ red, $\frac{3}{4}$ blue	6
3. $\frac{1}{3}$ red, $\frac{1}{3}$ blue, $\frac{1}{3}$ yellow	8
B. Colored Blocks (or other objects) Blocks of the same size differing only in color. Red, blue, green, yellow, orange.	5 of each color
C. Cubes (or dice)	
1. White, numbered 1 through 6	2
2. Red, numbered 1 through 6	2
D. Checkers Star on one side and crown on the other.	4
E. Cloth or Heavy Paper Bags About six to twelve inches deep and four to six inches in diameter.	4
F. Crayons	1 box
G. Cards Plain, white, three inches by five inches.	6
H. Oatmeal Box (or similar non-transparent box)	3
I. Coins	3
J. Felt Pens, red and black	2

	Number Needed
II. Optional	
A. Large Spinners for Classroom Viewing	
1. $\frac{1}{2}$ red, $\frac{1}{2}$ blue, numbered as the small spinners	1
2. $\frac{1}{4}$ red, $\frac{3}{4}$ blue, numbered as the small spinners	1
3. $\frac{1}{3}$ red, $\frac{1}{3}$ blue, $\frac{1}{3}$ yellow, numbered as the small spinners	1
B. Colored Chalk	1 box
C. Large Beads (or other objects) Alike except for color. To be used instead of colored blocks.	5 of each color
D. Newsprint, white wrapping, or construction paper	20 large sheets
E. Marbles 10 each of 6 colors, alike except for color	60
F. Paper Cups	
G. See suggested probability devices in the Appendix of the pupil text	
H. Liquid Duplicator Masters and Paper	
I. Overhead Projector and Materials	

MATHEMATICAL COMMENTS

1. The idea of probability.

Some events are regarded as certain: the sun will rise tomorrow, the coin will fall either heads or tails.

Other events are uncertain, depending upon chance fluctuations: it will rain tomorrow, the coin will fall heads.

In everyday conversation, people frequently compare the likelihood of (uncertain) events:

> They are all equally likely.
>
> This is more likely than that.
>
> I think it will rain this afternoon. (It is more likely to rain this afternoon than not to rain.)

In more scientific work, we compare likelihoods numerically. Every event is assigned a number describing or "measuring" its likelihood; the more likely the event, the higher the number assigned to it. The numerical measure of the likelihood of an event is called the probability of the event. The method of assigning probabilities to events and the study of the relations among them constitute the Theory of Probability.

A comment may be in order about the word "theory". In mathematics, the word is used to mean "body of knowledge". This is in contrast with its conversational meaning of "conjecture", as when we say, "The detective had a theory about who fired the gun." The theory of probability consists of our accumulated knowledge about probability. Similarly, mathematicians speak of the theory of numbers, the theory of groups, the theory of functions of a real variable, etc. No speculation is involved. Everything in a mathematical theory is true.

Modern science, industry, agriculture, and human affairs all depend strongly upon the theory of probability, either directly or via the theory of mathematical statistics which itself is based upon probability. The theory of probability underlies the biological laws of heredity; it is needed in fundamental research in physics and astronomy and, as a matter of fact, in working out plans for travel into space; it is used in developing and testing new drugs and medicines; it is the basis of agricultural experiments that lead to improved methods for the farmer; it guides the manufacturer in controlling the quality of his product, the industrial laboratory in the design of new equipment, the economist and psychologist in their studies of behavior, the military commander in his choice of tactics.

Probability theory is also the basis for analyzing games of chance. In fact, the theory of probability had its origins, about 300 years ago, in the gambling halls of Europe. Moreover, in explaining the theory and in working problems, it is necessary to toss coins and throw dice. Occasionally a person will conclude that to teach probability is immoral. He reasons: "Gambling is evil. Gamblers have to use probability. Therefore we should not teach probability." One might equally well reason: "Gambling is evil. Gamblers have to read and write. Therefore we should not teach reading or writing."

We teach probability not because of its possible misuse but because of its applications for the benefit of mankind.

A particularly important application is to mass behavior where each individual action is subject to chance fluctuations. For example, while the theory of probability will not tell us whether the next coin tossed will actually land heads, it does tell us that of the next 10,000 tosses the number of heads will lie, with virtual certainty, between 4500 and 5500. Despite the randomness, there is stability in the whole. The principle applies (with perhaps different numerical values) not only in genetics and the other pure sciences but to industrial and social phenomena as well: to manufactured items rolling off an assembly line, to a mass of vacationers deciding where to stop for lunch.

2. How probabilities are found.

The probability assigned to any event is, roughly speaking, the fraction of the time we expect it to happen. The probability of an event is, therefore, a number between 0 and 1, inclusive. An impossible event is assigned the probability 0. (It never happens.) A certain event is assigned the probability 1 (= 100%). An event that is just as likely to happen as not to happen is assigned the probability $\frac{1}{2}$; for example, if a coin is as likely to fall heads as tails (and will never land on edge), then the probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$.

Of course, most of the events we study are more interesting and more involved than these. How are their probabilities determined? There are two ways: by observation, and mathematically.

Probabilities that are determined by observation are of this sort:

> The probability that a child born today will live through its first year.
>
> The probability that a tossed coin will land heads.

The first of these is judged empirically, from birth-and-death records of recent years. The second is obtained by considering the symmetry of the coin and checking with some experimental tosses.

Probabilities that can be determined mathematically are of this sort:

> The probability that at least 85 of any 100 children born today will live through their first year.
>
> The probability that of 100 tossed coins, between 40 and 60 will land heads.

These complex events are made up of combinations of the simple events of a single child or coin, whose probabilities are assumed known (from observation). The probabilities of the complex events can be computed from those of the simple events according to the mathematical laws of probability.

We hope to give the child an appreciation of both these methods. We have him make judgments of empirical probabilities by performing actual experiments, recording their outcomes, and analyzing the results. We also guide him, informally, to some of the ideas underlying the mathematical formulas.

What is important is the spirit. We wish to develop in the child a feeling for the subject - an educated intuition. If he learns to make qualitative comparisons based on understanding, we have succeeded. If the best he can ever do is perform quantitative computations based on memorized rules, we have failed.

3. Comparisons of likelihood.

All children have had experience in making informal comparisons of likelihood, and this is a good topic to begin with.

> Example 1. Which is more likely on the 4th of July (here in our town) - rain, or snow?

This simple question leads to three interesting points for discussion.

(1) The comparison itself should be obvious to the children. Rain is more likely than snow because the 4th of July is usually too hot for snow.

(2) Notice that we can compare the likelihoods of two events even though we do not take into account all possible alternatives. There might be neither rain nor snow.

(3) The likelihoods being compared might both be very small. In many localities the chances for either rain or snow are virtually nil. But we can still compare the one with the other.

> Example 2. Billy is a very good student. Which is he more likely to do in tomorrow's test - pass, or fail?

Here again the comparison is evident: good students tend to pass, not fail. In this example there are no additional alternatives.

> Example 3. A new boy has joined the class. Mary is to guess his birthday. Alice is to guess how many brothers and sisters he has. Who is more likely to guess right?

This time we have no past experience to draw from. The answer is still easy, but we use a different method. Clearly, Mary has a wider selection to choose from and hence has more opportunities for guessing wrong. (We don't even have to count to see that.) Therefore, Alice is the one more likely to guess right. In this example, the most likely result is that they will both be wrong.

Events being compared may be related to each other in some way. In Example 1, rain and snow are related, in the sense that contributing causes favoring one also favor the other. Another relation between them is that of exclusion: they can't both happen at the same time (provided we define things right). Finally, it may turn out that neither one of them takes place. In Example 2, pass and fail are strongly related: one or the other must occur, but not both. In Example 3, the two guesses are not related at all. Mary's guess is of no help to Alice, nor Alice's to Mary. One guess could be right while the other is wrong; both girls might guess right; both might be wrong.

In each of these examples, it can happen that the less likely of the two events occurs while the more likely one does not. It can snow. Billy might fail. Mary may get the birthday right even though Alice is wrong about the family.

> Example 4. When Mary guesses the new boy's birthday, which (in her guess) is the more likely to be right - the entire birthday, or the month alone?

This example is fundamentally different from the others and illustrates an elementary but important principle. If the boy was born on January 14th, then he was born in January. It is impossible for the less likely event (hitting the birthday) to occur while the more likely (hitting the month) does not. In this case, the comparison in likelihood is derived not from past experience or numerical considerations, but from logical necessity.

4. Estimating probabilities by experiment.

The probability of an event is, roughly speaking, the fraction of the time we expect it to happen. An estimate, then, is the fraction of the time it did happen (under similar conditions). Records of births over a period of years show that the fraction of boys is .5 (to within two one-hundredths); we conclude that the probability is .5 that the Jones's next child will be a boy. A major league player has a batting average of .303; we conclude that the probability of a hit his next time at bat is .303.

Children will be less interested in tables of actual statistics than in performing experiments for themselves. An excellent project is to estimate a probability about which they have no preconceived notion; for example, the probability that a thumbtack, when jiggled and thrown onto the table, will come to rest on its head. (If tacks are too hazardous, substitute rivets, bottle tops, etc.)

Our estimate of the probability is the number of heads divided by the number of throws. How many throws should be made? Questions like this can be subjected to delicate analysis, as part of the theory of statistics. Our own discussion will be much more elementary.

Before plunging into the experiment with the tacks, let us gain some insight by experimenting with the humble coin. We assume that the coin is "honest" - that is, the probability of heads is $\frac{1}{2}$.

By the way, a fair toss or spin requires some skill beyond many children. A little practice may be worth while even for the teacher. If the experiment calls for a large number of throws, and if you are not keeping track of the order of the results but only the totals, throw the coins 10 at a time. The probabilities are the same whether the throws are successive or simultaneous. Not only do you speed things up, but the whole affair is easier to control: mix by shaking in your cupped hands, and release gently onto the table. Cover the table with paper to cut down on noise. (With some modifications, you can keep track of order as well. For example, record the results of 4 coins in the order penny, nickel, dime, quarter.)

Back to our coin. Let us toss it 10 times. (Or toss 10 coins at once.) Since the probability of heads is $\frac{1}{2}$, we expect heads $\frac{1}{2}$ the time in many tosses. So we expect heads 5 times out of 10 in the long run. But of course we cannot count on getting exactly 5 heads in our particular throw of 10.

Actually, the most likely result in 10 throws is indeed to get exactly 5 heads. But this result is not likely! It will happen only about 1 time in 4. (The methods for calculating these probabilities will be gone into later on.) This fact will come as a surprise to most children. But confronting them with the theoretical quotation is not enough. For the fact to soak in, they should perform (or observe) some actual experiments.

Let us assume that in our throw of 10 we counted 7 heads. This is a perfectly reasonable result. (1 time in 3 we will get at least 7 heads or at least 7 tails.) If we stop here, our estimate of the probability of heads is .7 - pretty far off.

But suppose we throw 100 times. To obtain as many as 70 heads would be highly unusual. A reasonable result would be something between 43 and 57, say. Our estimate would now be sensibly near the true value.

Suppose we throw 1000 times. To obtain as many as 700 heads would be a miracle. Even a result like 430 or 570 is highly improbable. A reasonable result would be something between 475 and 525, say. Our estimate would now be quite close.

And now to the tacks. We throw 10 of them and count 7 on their heads, 3 on their sides. Is the probability of heads .7 ? That would be a rash conclusion. As we have just seen, we might well get the same count when the probability is actually .5 . So we make another throw of 10 . And another.

We don't know just when it is safe to stop. In fact, there is no way of knowing when to stop. But we notice after a while - perhaps only after two or three hundred trials - that the results start to settle down. We watch the small random fluctuations get swallowed up in the overwhelming stability of the mass. Then we announce our estimate in a firm voice.

5. *Equally likely outcomes.*

Let us throw a die which, we are assured, is honest - that is, any one of the six faces is as likely to show as any other. Then each face will appear, in the long run, $\frac{1}{6}$ of the time. Hence the probability of any particular face, on a given throw, is $\frac{1}{6}$. The probability of a 4, for instance, is $\frac{1}{6}$.

This example illustrates a general principle. Suppose that an experiment can result in any one of a number of outcomes, *all equally likely*. Call the total number of possible outcomes n. Then the probability of any particular outcome is $\frac{1}{n}$.

In the example, the experiment is the throw of the die; the possible outcomes are the faces: 1, 2, 3, 4, 5, 6, *all of which are equally likely*; their number, n, is 6; and the probability of any particular outcome, $\frac{1}{n}$, is $\frac{1}{6}$.

In throwing a die, what is the probability of a number greater than 4 ? In other words, what is the probability of throwing either 5 or 6 ? Each of these appears (in the long run) $\frac{1}{6}$ of the time. Therefore one or the other appears $\frac{2}{6}$ of the time - that is, $\frac{1}{3}$ of the time. So the probability of a number greater than 4 is $\frac{1}{3}$.

The principle here is an extension of the one above. Suppose, again, that an experiment can result in any one of a number of outcomes, *all equally likely*; call the total number of possible outcomes n. Certain of the outcomes are "favorable" to the event we are interested in, while the rest are not; that is, the event takes place if one of the favorable outcomes occurs, but not otherwise. Call the number of favorable outcomes s. Then the probability of the event is $\frac{s}{n}$. (The earlier principle handles only the special case in which $s = 1$.)

In words: the probability of the event is equal to the number of favorable outcomes divided by the total number of possible outcomes.

In the example, the experiment is the throw of the die; the possible outcomes are 1, 2, 3, 4, 5, 6, *all equally likely*; $n = 6$; the event of interest is a number greater than 4; the favorable outcomes are 5 and 6; there are two favorable outcomes, so s is 2; and the probability of the event $\frac{s}{n}$ is $\frac{2}{6}$, that is, $\frac{1}{3}$.

It is instructive to test these conclusions by actual experiments. Make a large number of throws and check that a number greater than 4 comes up about $\frac{1}{3}$ of the time.

We have described two methods for obtaining probabilities: empirically, and by counting equally likely outcomes. In applications, it is sometimes hard to tell whether various outcomes really are equally likely. This problem, discussed in the next section, will lead to the question of how known probabilities are combined to obtain new ones.

6. *Probabilities of combinations.*

Suppose that we are interested in the number of heads showing on a throw of two coins, say a penny and a dime. There may be none at all, or just 1, or 2. Are these three outcomes equally likely? If you are in any doubt at all about the answer, make 50 throws and see what happens.

It is a common pitfall simply to assume that outcomes presented in a natural way are equally likely. The safeguard is to consider them carefully and supply a reason in support of the assumption.

The danger in our problem is to be swindled by one's own language. Let us replace the dime by a checker. Even better - keep the dime, but just *say* "star" for its heads and "crown" for its tails. On a throw of the two coins there are 4 possible outcomes: heads-star, heads-crown, tails-star, tails-crown; *and there is not the least doubt that these 4 outcomes are all equally likely*. Their original names are heads-heads, heads-tails, tails-heads, tails-tails (the first word in each case referring to the penny, the second to the dime). It is now clear that the probability of 2 heads is $\frac{1}{4}$, the probability of just 1 head is $\frac{1}{2}$, and the probability of 0 heads is $\frac{1}{4}$.

What is the situation with 3 coins - say a penny, a nickel, and a dime? For instance, what is the probability that exactly 2 fall heads? Each coin will fall in either of 2 ways, independently of how the other two fall. Altogether then, the 3 coins can fall in $2 \times 2 \times 2 = 8$ possible ways, *all equally likely*. They are:

HHH;   HHT, HTH, THH;   HTT, THT, TTH;   TTT

Here H denotes heads and T tails; the first letter in each triple refers to the penny, the second letter to the nickel, and the third to the dime. In this list, we have grouped the results as follows: first, the outcome consisting of 3 heads; next, the outcomes with just 2 heads; then those with just 1 head; then the one with 0 heads. There are 3 with just 2 heads. Therefore the probability of getting exactly 2 heads is $\frac{3}{8}$: the number of favorable outcomes divided by the total number of outcomes.

This is a good time to pause for experiment. Take 3 pennies, throw them 20 or 30 times, and see what happens.

Looking back, it should be clear why the number of favorable outcomes (those yielding exactly 2 heads) is 3. That is precisely the number of ways we can select 2 coins (for heads) from the 3 coins: as penny-and-nickel (HHT), penny-and-dime (HTH), nickel-and-dime (THH). Likewise, just 3 of the possible outcomes yield exactly 1 head: penny (HTT), nickel (THT), dime (TTH). Finally, there is just 1 way (HHH) to pick 3 heads, and just 1 way (TTT) to pick 0 heads.

The reasoning in the general case is similar. Suppose we toss n coins. Each one will fall in either of 2 ways, independently of how the others fall. Altogether, then, the n coins can fall in

$$\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ factors}} = 2^n$$

possible ways, *all equally likely*. Consider any number s between 0 and n, inclusive. How many of the 2^n outcomes yield exactly s heads? Answer: the number of ways of selecting s coins (for heads) from n coins. The symbol for this number is $\binom{n}{s}$:

$$\binom{n}{s} = \text{number of ways of selecting } s \text{ objects from } n \text{ objects.}$$

The probability of obtaining exactly s heads on the throw of n coins, is, then $\frac{\binom{n}{s}}{2^n}$: the number of favorable outcomes divided by the total number of outcomes.

The symbol $\binom{n}{s}$ is read "n above s" (not "n over s"). It is not a fraction, even though it looks something like one. It is a special symbol, the one most popular among mathematicians for the number of ways of selecting s objects from n objects. In place of $\binom{n}{s}$, some writers use $C(n,s)$ or $C_{n,s}$ ("the number of Combinations of n things taken s at a time"). None of the symbols is used in this teaching material.

7. Computing the number of combinations.

We have to know how to compute $\binom{n}{s}$ for various values of n and s. When n and s are small, the computations are easy. For instance, the example of the 3 coins shows that $\binom{3}{2} = \binom{3}{1} = 3$, and that $\binom{3}{3} = \binom{3}{0} = 1$.

It is clear that $\binom{1}{1} = 1$, $\binom{2}{2} = 1$, and so on: $\binom{n}{n} = 1$ whatever the value of n; for, there is just 1 way of selecting n objects from n objects (namely, choose them all). It is also easy to see that $\binom{1}{0} = 1$, $\binom{2}{0} = 1$, and so on: $\binom{n}{0} = 1$ for every value of n; for, there is just 1 way of selecting 0 objects from n objects (namely, omit them all).

Next, we see that $\binom{1}{1} = 1$, $\binom{2}{1} = 2$, $\binom{3}{1} = 3$, and so on: $\binom{n}{1} = n$ whatever the value of n; for, there are just n ways of selecting a single object from n objects (namely, pick any one of them). There are also just n ways of picking $n-1$ objects from n objects (namely, omit any one of them); therefore $\binom{1}{0} = 1$, $\binom{2}{1} = 2$, $\binom{3}{2} = 3$, and, in general, $\binom{n}{n-1} = n$.

For the next observation, consider as an illustration the case $n = 10$, $s = 8$. (Suppose, for example, that we are interested in the number of ways 10 coins can show exactly 8 heads.) For each selection of 8 coins for heads, there remain 2 showing tails. A different choice of 8 heads yields a different remaining set of 2 tails. If we run through all possible selections of 8 coins, then (from those left over in each case) we simultaneously run through all possible selections of 2 coins. It follows that there are exactly as many ways of choosing 8 coins from 10 as of choosing 2 coins from 10. Therefore $\binom{10}{8} = \binom{10}{2}$. Similarly, $\binom{10}{7} = \binom{10}{3}$, and $\binom{10}{6} = \binom{10}{4}$. The reasoning here is general and yields the general formula $\binom{n}{n-s} = \binom{n}{s}$.

The values of $\binom{n}{s}$ increase surprisingly rapidly. For example, $\binom{52}{5} = 2{,}598{,}960$, and $\binom{52}{13} = 635{,}013{,}559{,}600$. (These are the number of possible 5-card hands and the number of possible bridge hands, respectively.)

Small values of $\binom{n}{s}$ are easy to calculate by means of the so-called "Pascal triangle", named for the mathematician Blaise Pascal (1623 - 1662). The triangle starts out

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
 & & & & & \searrow & & \swarrow & & & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

representing the values of

$$\begin{array}{ccccccccccc}
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
 & & & & & \searrow & & \swarrow & & & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

and can be continued indefinitely.

The arrows shown illustrate the method by which new values are generated. Each entry (if not the first or last in its row) is equal to the sum of the two nearest it in the line above. In the illustration, $6 + 4 = 10$, representing $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$.

To see why this works, think of a man with his 4 sons on a camping trip. From the 5, a team of 3 are to be chosen to gather kindling. The number of possible teams is, then, $\binom{5}{3}$. Now, either Dad is on the team, or he isn't. If Dad is on the team, then his 2 partners are chosen from among the 4 boys; there are $\binom{4}{2}$ ways of choosing them. If Dad is not on the team, then the whole team of 3 is chosen from among the 4 boys; there are $\binom{4}{3}$ ways of choosing them. Altogether, then, there are $\binom{4}{2} + \binom{4}{3}$ ways of choosing the team. Consequently, $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$.

A formula for computing $\binom{n}{s}$ is given at the end of the next section.

One way to check one's arithmetic, when calculating from the triangle, is to see that the sum of the entries in the n^{th} row is equal to 2^n. For example, the sum in the 5^{th} row is

$$1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5.$$

This must be so because, as we saw on page 14 there are 2^n possible outcomes altogether in a throw of n coins; and in adding across the n^{th} row, we are counting up these same outcomes (the number with 0 heads, + the number with 1 head, and so on).

By the way, the entries in the Pascal triangle arise in algebra when one multiplies out an expression like $(H + T)^5$. The result is:

$$(H + T)^5 = H^5 + 5H^4T + 10H^3T^2 + 10H^2T^3 + 5HT^4 + T^5 .$$

The coefficients - 1, 5, 10, 10, 5, 1 - are precisely the entries, in order, in the 5^{th} row of the triangle. Moreover, the exponents show the numbers of occurrences. For example, the term $10H^3T^2$ gives us the information that there are 10 ways of getting 3 heads and 2 tails (in a throw of 5 coins).

8. Permutations.

A permutation is an ordered arrangement. One often finds it necessary to know how many permutations are possible of a given number of objects. For 2 objects - say the digits 1, 2 - there are just 2 permutations: 12, and 21. For 3, there are 6 permutations:

123, 132; 213, 231; 312, 321.

For example, there are 6 ways that Jim, Dick, and Sue can be assigned to clear the table, wash the dishes, and dry the dishes (one to a task).

The number of permutations of 3 objects is 6 because 6 is equal to 3 × 2 × 1. (We could write 3 × 2, but 3 × 2 × 1 looks better.) The reasoning goes like this. There are 3 choices for the first position. With each such choice, there are 2 ways of permuting the 2 remaining objects (in the 2 remaining positions). Hence there are 3 × 2 permutations of all 3.

The number of permutations of 4 objects is $4 \times 3 \times 2 \times 1 = 24$. For, there are 4 choices for first position. With each such choice, there are $3 \times 2 \times 1 = 6$ ways of permuting the 3 remaining objects (in the 3 remaining positions). Hence there are $4 \times 3 \times 2 \times 1$ permutations of all 4.

Similarly, the number of permutations of n objects is equal to the product of all the numbers from n down to 1. This number is known as "n factorial", and is denoted by $n!$ For example, $1! = 1$ and $2! = 2 \times 1 = 2$. Also, as we have seen,

$$3! = 3 \times 2 \times 1 = 6$$

and

$$4! = 4 \times 3 \times 2 \times 1 = 24.$$

These expressions, as well as the reasoning that led to them, suggest how to compute each new factorial from the preceding one. In computing $4!$, for example, we don't have to multiply out $4 \times 3 \times 2 \times 1$ if we already know that $3! = 3 \times 2 \times 1 = 6$; instead, we say, more simply,

$$4! = 4 \times 3! = 4 \times 6 = 24.$$

The saving becomes clearer as soon as one deals with larger numbers. Here are the next few factorials:

$$\begin{aligned}
5! &= 5 \times 4! = 5 \times 24 = 120; \\
6! &= 6 \times 5! = 6 \times 120 = 720; \\
7! &= 7 \times 6! = 7 \times 720 = 5{,}040; \\
8! &= 8 \times 7! = 8 \times 5{,}040 = 40{,}320; \\
9! &= 9 \times 8! = 9 \times 40{,}320 = 362{,}880; \\
10! &= 10 \times 9! = 10 \times 362{,}880 = 3{,}628{,}800.
\end{aligned}$$

As is seen, the numbers $n!$ increase more rapidly than most people might guess. Suppose that 10 children come up to sharpen their pencils. There are more than three-and-a-half million ways of deciding which child will be first, which second, and so on, down to which will be last! (No wonder we use an exclamation point!)

Factorials can be used to compute the numbers $\binom{n}{s}$. The formula is:

$$\binom{n}{s} = \frac{n!}{s!(n-s)!} \quad .$$

For example,

$$\binom{6}{2} = \frac{6!}{2!4!} \;, \qquad \binom{6}{3} = \frac{6!}{3!3!} \;,$$

and so on. In the actual calculations, there is always some cancelling that can be done first to simplify the work:

$$\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2}}{2 \times \cancel{4} \times \cancel{3} \times \cancel{2}} = \frac{6 \times 5}{2} = 15 .$$

There is one detail that sometimes looks strange at first. So far, we have not said what 0! is. The definition given above for 1! , 2! , and the others does not work for 0! ; therefore 0! needs a separate definition. The definition given is: 0! = 1 . The reason for this is to make the formula for $\binom{n}{s}$ work in every case. For example:

$$\binom{5}{0} = \frac{\cancel{5!}}{0!\cancel{5!}} = \frac{1}{0!} = 1 .$$

9. Yes or no.

In these concluding sections we describe some of the basic rules of probability.

We shall use the letter P as an abbreviation for "the probability of". In the place of "the probability of heads", for instance, we will write

$$P(\text{heads}) .$$

(The parentheses are there to help the eye.)

Suppose we wonder whether Billy will pass or fail tomorrow's test. He may be more likely to do one than the other, depending on circumstances. But in any case, two things are clear: he can't do both; and he must do one or the other. The the following is true:

$$P(\text{pass}) + P(\text{fail}) = 1 .$$

The rule is this. An event either happens or does not happen. We must have one or the other, and cannot have both. Then

$$P(\text{it happens}) + P(\text{it does not happen}) = 1 ;$$

more briefly,

$$P(\text{yes}) + P(\text{no}) = 1 .$$

In a typical problem, we know one of the values on the left (the first, say) and solve for the other:

$$P(\text{no}) = 1 - P(\text{yes}) .$$

For example, if the probability that Billy passes is .6, then the probability that he fails is .4 . If the probability that Dad takes Jim to the circus is .3, then the probability that he doesn't is .7 .

10. Either this or that.

We often consider alternatives that exclude one another but do not necessarily include all possibilities. For example, Dad may take Jim to the circus; or, if not, he may take him to the ball game; but he won't do both. (And they might stay home.) Suppose that

$$P(\text{circus}) = .3$$

and that

$$P(\text{game}) = .1 .$$

What is the probability that Jim will get to one or the other?

Answer:

$$P(\underline{\text{either}}\text{ circus }\underline{\text{or}}\text{ game}) = P(\text{circus}) + P(\text{game})$$
$$= .3 + .1 = .4 .$$

The rule is as follows.

$$P(\underline{\text{either}}\text{ this }\underline{\text{or}}\text{ that}) = P(\text{this}) + P(\text{that})$$

provided the alternatives exclude each other.

There is also a rule for the case of alternatives that do not exclude each other, but it is more complicated, and we shall not go into it. The examples that arise in this teaching material can all be handled by counting the possible outcomes.

Suppose, for instance, that we play a game by throwing a die. Ann scores a point if it shows 1, 2, or 3. Betty scores a point if it shows 2, 4, or 6. Then the probability that Ann scores is $\frac{3}{6} = \frac{1}{2}$ and the probability that Betty scores is also $\frac{3}{6} = \frac{1}{2}$. To find the probability that someone scores - either Ann, or Betty, or both - simply count the favorable outcomes. There are 5 favorable outcomes: 1, 2, 3, 4, 6; and all are equally likely. The total number of possible outcomes is 6. Therefore the probability that someone scores is $\frac{5}{6}$.

11. *Both this and that.*

Mother announces that the probability of having ice cream for dessert is .8 ; and furthermore, that this is irrespective of whether Jim goes to the circus, goes to the ball game, or stays home. What is the probability that Jim both goes to the circus and gets ice cream? Answer:

$$\begin{aligned} P(\underline{\text{both}}\text{ circus }\underline{\text{and}}\text{ ice cream}) &= P(\text{circus}) \times P(\text{ice cream}) \\ &= \ .3 \quad \times \quad .8 \\ &= \ .24 \ . \end{aligned}$$

The rule is:

$$P(\underline{\text{both}}\text{ this }\underline{\text{and}}\text{ that}) = P(\text{this}) \times P(\text{that})$$

provided that the two outcomes are independent of one another.

Independence means that the probability of either is unaffected by the occurrence of the other. Sometimes the question is hard to decide, but the examples in the teaching material should offer no difficulty.

Here is an example where the outcomes are not independent. We have a jar containing 3 marbles: 1 red, 1 white, and 1 blue. From it we draw a marble at random, and then, *without replacing it*, we draw another. We are interested in the following results:

red on the first draw,

and

blue on the second draw.

Surely, the probability of the latter is affected by the occurrence of the former. Let us check.

The probability of blue on the second draw is $\frac{1}{3}$, as can be seen by either of the following methods.

(1) There are 3 possible outcomes of the second draw (R, W, B). All are equally likely, and 1 of them (B) is favorable.

(2) There are 6 possible outcomes of the sequence of drawings (RW, RB, WR, WB, BR, BW). All are equally likely, and 2 of them (RB, WB) are favorable.

On the other hand, if the first event does occur - if the first draw does yield red - then the probability of blue on the second draw becomes $\frac{1}{2}$. For, now, there are just 2 possibilities (B, W), and just 1 of them (B) is favorable.

Since the probability of blue on second $\left(\frac{1}{3}\right)$ is different from the probability of blue on second if red occurred first $\left(\frac{1}{2}\right)$, the two events

red on first, blue on second

are not independent.

LESSON 1

Thinking About Chance

Introduction.

While all children are aware that chance and uncertainty are a part of daily life, the conclusions they reach as a result of this awareness vary widely. The frequency with which they attribute the outcome of chance events to luck or to "the law of averages" provides ample evidence that many of them have not learned to analyze situations in a thoughtful way before forming conclusions.

The purpose of this lesson is to stimulate pupils to think more objectively about chance events. Through participation, discussion, and demonstrations by the teacher, pupils will have opportunities to test their intuition or hunches regarding the results of some activities involving chance and to make guesses, estimates, and predictions about such results.

The Pupil Exercises in this and other lessons give opportunities to use some of the ideas developed in the lesson. They also provide some optional activities which may be completed outside regular class time by interested pupils.

The lesson is developed informally and pupils should be encouraged to state their ideas and conclusions in their own words rather than in any formal mathematical terms. Unanimity of opinion is not a goal but pupils should be encouraged to test their opinions and hunches by analyzing the situations that gave rise to them.

The Suggested Procedure in this and the following lessons provides a sequential development of the pertinent ideas in the lesson. However, it is not intended to be prescriptive and you should feel free to paraphrase, supplement, or disregard it as seems appropriate in a particular situation. On many of the lessons you will find that the use of the overhead projector is most helpful.

Vocabulary: Chance, certain, uncertain, more likely, outcome, possible, impossible, possibility, probable, probability.

Materials: (Also used in Lesson 2)

(1) Three spinners with dials of $\frac{1}{2}$ red, $\frac{1}{2}$ blue; $\frac{1}{4}$ red, $\frac{3}{4}$ blue; $\frac{1}{3}$ red, $\frac{1}{3}$ blue, $\frac{1}{3}$ yellow. (These spinners come with this unit.)

(2) Two or three coins.

(3) Two dice.

(4) Two sets of three cards, two inches by three inches; in each set a triangle is drawn on two of the cards and a square on the third card.

(5) Two sets of two cards, two inches by three inches; in each set a triangle is drawn on one card and a square on the other.

(6) Six objects differing only in color, such a cubes, checkers, beads, new crayons, pegs, marbles, etc., so that these sets can be obtained:

  (a) Three red objects and one yellow object,

  (b) one red object and one yellow object.

  (Colors other than red or yellow could, of course, be used.)

  (c) A box or non-transparent sack to hold these objects. (Large cloth bags about twelve inches deep and six inches in diameter would be very useful.)

Suggested Procedure:

After an appropriate statement such as, "Now, for a while we will be doing some different things in our mathematics classes," ask a pupil to come to the front of the room.

one red cube
and one
yellow cube

Use Materials (6). As you put a red cube and a yellow cube in the sack or box, ask the class to guess which cube the pupil will pick from the sack.

Discuss their expectations. They'll probably make statements such as:

> "It's fifty-fifty." "Chances are even." "It's equal either way." "It's one out of two." "Who can tell?" "You can't be sure." "He's as likely to get red as yellow."

Comment that these statements are concerned with chance; that there are many events in life which are uncertain and this is one of them. Discuss what they might expect if each of them came up to draw a cube or if John chose several times, returning the cube drawn to the sack each time.

You might ask them to suggest other things in life which are uncertain; for example:

> Who will win the World Series?
> Will I have ice cream for lunch some time this week?
> Will all the pupils in the class be at school next Monday morning?

Ask pupils to suggest things that are certain. For example, day follows night; there are twelve inches in a foot. Then suggest things which are impossible. From their statements, draw out the idea that some things are more likely to happen than others and that some things are more likely to happen than not. Pupils can respond to such questions as:

> Which is more likely, that some pupil will be absent or that the teacher will be absent?
> Which is more likely, that you will have cereal for breakfast or that you'll have cereal for lunch?
> Which is more likely, that a boy will build a model airplane or that a girl will build a model airplane?

Now show the class that you are putting three red cubes and one yellow one in the box, and ask another pupil to draw out one cube which will be replaced in the box. Pupils can guess again which color they expect to be drawn. Let them discuss reasons for their expectations.

three red cubes and one yellow cube

> If everyone in the class took turns drawing a cube, what would we expect might be the outcome. Would red be chosen more often than yellow? (We would expect it to be, although it is possible, but not very probable, that red would not be drawn at all.)

Show the spinner with the dial $\frac{1}{4}$ red and $\frac{3}{4}$ blue. Ask where they have seen spinners before, for what they were used, and why they were used. Bring out that spinners select "by chance" because the person who spins it cannot know in advance where it will stop.

spinner $\frac{1}{4}$
red and
$\frac{3}{4}$ blue

> If Mary spins this spinner, will the arrow point to red or to blue?

Let them discuss their expectations and their reasons; then raise such questions as:

> Can you be sure it will point to red? to blue?
> Will it point either to red or to blue? (Chances are that it will. However, it could stop on a line.)

Mary spins and the pupils compare their expectation with the outcome. The spinner should be horizontal rather than vertical so that the pointer is not biased. The "fairness" of the spin could be noted, for example:

> Would it make a difference if Mary spun the pointer in the other direction? If she pushed the pointer harder? If the pointer were pushed nearer the head? nearer the tail? (Develop the idea that the spinner is honest.)

Ask the class what they would expect if Mary spun the spinner 100 times. (It would stop on blue more often than on red--not exactly outcomes of 75 blue and 25 red but something close to this. Or, it will stop on red less often than on blue. It is not expected that pupils will suggest these numbers but they should see that the expectation of blue is greater than of red.)

spinner $\frac{1}{3}$ red
$\frac{1}{3}$ blue, and $\frac{1}{3}$
yellow

Show the spinner with the dial $\frac{1}{3}$ red, $\frac{1}{3}$ blue, and $\frac{1}{3}$ yellow. Ask pupils what they think will be the outcome of one spin and other questions similar to those asked before. Do not indicate whether their answers are correct.

> How many different possible outcomes are there? (Three, assuming the arrow doesn't stop on a line.) Is there a better chance that the outcome will be blue on one spin with this dial or on one spin with the $\frac{1}{2}$ red and $\frac{1}{2}$ blue dial? (Show both dials.) What would you expect the outcomes to be on 100 spins?

spinner $\frac{1}{2}$
blue and
$\frac{1}{2}$ red

Again show the spinner with the dial $\frac{1}{2}$ red and $\frac{1}{2}$ blue. Ask what their expectations would be on one spin; on 100 spins.

In what way is spinning this spinner similar to the activity of drawing one block from the sack which held one red cube and one yellow cube?

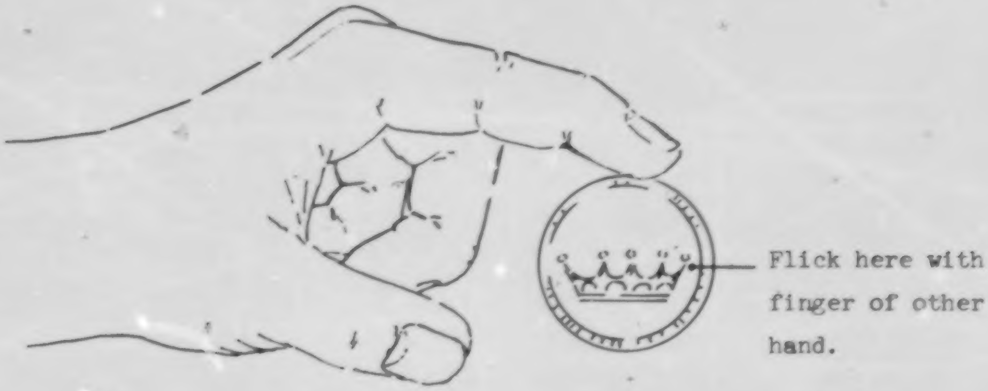
coin

Show the coin, preferably a half-dollar because it is easy for the children to see.

If I toss or spin this coin, will it land with the "head" up?

Discuss which is heads and which is tails; what constitutes a fair toss or spin; that we'll assume the coin doesn't land on its edge so there can be only two possible outcomes--heads or tails; that the coin is honest. In thinking about honest coins, you should be aware that the 1964 penny is NOT honest. If it is spun, on a truly smooth surface, it will land tails up more often than heads up.

Since a penny is lighter than a half-dollar, is it more likely to come up heads than the half-dollar? (In tossing any honest coin, a head is just as likely as a tail.)


Flick here with finger of other hand.

Demonstrate how to toss or spin the coin. Some children find it easier to shake the coin in a paper cup. If a paper is placed on the desk, the noise will be reduced. Let children guess the outcome. Comment that we'll be doing activities using coins.

> Is tossing a coin related to spinning the spinner with the dial $\frac{1}{2}$ blue and $\frac{1}{2}$ red? to drawing one of the two blocks in the sack?

Children will probably see the relationship--"all are equal chances", "one out of two", etc.

die

Show the die, mentioning that it is a cube. Raise questions such as: How many faces does it have? What numbers do the dots represent? If I toss it, is there a fifty-fifty chance that the two dots will show on the top face? What do you think will show when I toss it once?

Toss the die; note that the outcome is the face that is up.

> Do you suppose if I toss the die again, the outcome will be the same?
>
> Would you expect a better chance for a three than a one?
>
> Is a six more likely than a four?

Let pupils discuss their expectations but do not tell them whether or not their ideas are correct. Some pupils may see that each face has an equal chance to be up.

> If a king told you that he'd give you a sack of gold if you could get one of the following outcomes, which would you choose?
>
> 1. Blue on a spinner whose dial is $\frac{1}{2}$ red and $\frac{1}{2}$ blue.
> 2. A 2 on one toss of a die.

Let pupils discuss their choices and the reasons for them.

At this point you might like to use two dice to play a game with the class. Ask them what the sum of the dots on the two faces that are up can be (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) and write these sums on the board. Suggest that you'll toss the dice and if the sum is 2, 3, 4, 10, 11 or 12, the class gets one point. If the sum is 5, 6, 7, 8 or 9, you get one point. Show them that they win on six sums and you win on only five sums. Toss the dice and keep score on the board. You have twice as many chances to win as the class. This can be seen from this table of sums. For example, there are six different ways to obtain a sum of 7 but only one way to obtain a sum of 2 or of 12. Out of 36 possible sums, you win on 24 of them and the class wins on only 12. Do not explain why you are more likely to win but encourage the class to express their ideas. Mention that in this unit they will learn many interesting things about probability.

First die \ Second die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

cards

I am a very good guesser. I can sense what is on the other side of these cards. (Put the two cards with the triangle on them face down on your desk.) I will now select the card with the drawing of a triangle on it. (Do so and the pupils will sense a trick and want to see both cards. Show these and then add the card with the drawing of the square on it.) Ask questions such as:

On one draw, would you expect to draw a card with a triangle on it or with a square on it?

Is there a better chance of drawing a card with a triangle or the card with the square on it?

Pupil pages 1-3: These pages summarize and serve as a record of the class discussion. They can be discussed rather briefly.

Pupil pages 4-6: The exercises can be completed independently and then discussed as a class activity. Encourage the pupils to do all or a part of the optional section, Things To Do At Home. These completed activities should be discussed in class.

The activity on page 6 has four possible outcomes; heads on the first penny and heads on the second penny; heads on the first penny and tails on the second penny; tails on the first and heads on the second; tails on the first and tails on the second. These are usually written--HH, HT, TH, TT. So the probability of two heads is $\frac{1}{4}$(HH), the probability of a head and a tail is $\frac{1}{2}$(HT,TH) and the probability of two tails is $\frac{1}{4}$(TT). These probabilities would not be discussed with the class at this early point in the unit.

1. THINKING ABOUT CHANCE

You probably have heard or even made statements such as these:

1. More likely than not we will go to the park on Saturday.
2. Chances are good that we will get to do it.
3. John and Billy have equal chances to win.
4. I am almost certain that I can come to your house after school.

These sentences are alike in one way. They have words and ideas which are used in mathematics. These words and ideas are used in a part of mathematics called probability. In probability, we are interested in things which happen by chance. By using mathematics we can often estimate quite accurately what will probably happen.

We will experiment with such things as coins, spinners, colored blocks, and dice to learn what to expect. Later we will learn what to expect by working with numbers instead of using experiments.

What Do You Know About Chance?

Do you know the answers to these questions?

Who will win the World Series this year?

Will all the members of our class be in school next Monday?

How many people in our class will have perfect spelling papers this week?

Will I see a Ford truck on my way home from school this afternoon?

We cannot be certain of the answers to questions such as these because they are chance events. However, there are some things about chance events which we do know.

Some things are more likely to happen than others. For example:

Which is more likely, that one of the pupils will be absent or that the teacher will be absent?

Which is more likely, that you will have cereal for breakfast or that you will have cereal for lunch?

Which is more likely, that a boy will build a model airplane or that a girl will build a model airplane?

Some things are more likely to happen than not. Think of answers to these questions:

In Phoenix, Arizona, in July, is it more likely than not that the sun will be shining at noon?

Is it more likely than not that you can find the sum of 324 and 465 ?

Is it more likely than not that your neighbor has
a TV set?

Some things are certain and some things are impossible. Which are these?

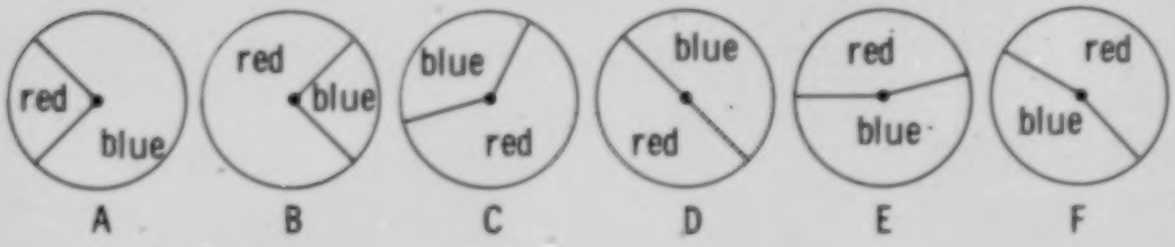
A man can live without any liquid for three months.
I will use my brain some time this week.
My dog can write his first and last name in Russian.
All new cars this year will use water for fuel.
Tomorrow, today will be yesterday.

Our ideas about chance might be classified Certain, Uncertain, or Impossible. In front of the following sentences, write C, U, or I for Certain, Uncertain, or Impossible.

I 1. Sun will set in the east.
C 2. A river flows downhill.
U 3. We will see the sun today.
C 4. Sun will rise
I 5. A river flows uphill.
U 6. A river is deeper today than yesterday.
U 7. I will sleep 8 hours on Monday.
C 8. I will sleep sometime this week.
I 9. I will not sleep at all this week.

Exercises - Lesson 1.

Use these pictures for Exercises 1 through 3.


red
blue
A
red
blue
B
blue
red
C
blue
red
D
red
blue
E
red
blue
F

1. Circle the letter of the spinner whose pointer is more likely to stop on red than blue.

(a) A or [✓ B]?
(b) [✓ C] or D ?
(c) E or [✓ F]?

(d) [✓ B] or C ?
(e) [✓ D] or E ?
(f) [✓ C] or F ?

2. Study spinner D and answer these questions.

(a) Could you get 100 reds in 100 spins on this spinner? yes

(b) Are you likely to get 100 reds in 100 spins on this spinner? no

(c) About how many reds do you expect from 100 spins? 50

3. Suppose a pirate captain said to you, "I will give you just one chance on a spinner. If the pointer stops on blue, into the sea you go. If it stops on red, you may go free."

   (a) If the captain let you choose one of these six spinners, which would you choose for your chance? B

   (b) If the captain allowed you to make the spinner, how would you color the dial? all red

   (c) If the captain were very angry, how do you think he would color the dial? all blue

Things To Do At Home - Lesson 1.

1. Look for stories in the newspaper that use some of these words.

   probable
   chance
   likely

   probability
   equal chance
   unlikely

   Bring them to share with your class.

2. Place a newspaper on a table. Shake two pennies in your cupped hands and drop them on the paper. Make a tally mark on a piece of paper to show the number of heads or tails. (See the example below.)

2 heads	1 head and 1 tail	2 tails
𝍸 𝍸 ||	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |||	𝍸 𝍸 𝍸

Do this 60 times and record the results in a table like the one above.

1. How many times did you get heads on both coins? ________
2. How many times did you get tails on both coins? ________
3. How many times did you get one head and one tail? ________
4. Are your answers to the first three questions the same number? ________
5. Are any two of your results equal? ________
6. Do you expect to get fewer "2 heads" than "1 head and 1 tail" ? ________

LESSON 2

Using Activities to Learn About Chance

Introduction.

In the first lesson children participated primarily through discussion. In this lesson, they will become more active participants by working together in teams or committees. Their experience with the various devices should help them build an educated intuition concerning chance events and get more of a feeling for probability.

The ten different activities suggested in this lesson can best be accomplished through committee work, with several committees completing each activity. This really serves two purposes: First, the pupils are actively engaged in experimentation and second, more data are obtained.

Because each activity is carried on for only 50 trials, it is possible that a committee might report outcomes quite different from what we would expect. This is really a tremendous opportunity. Pupils should learn, for example, from Activity 2, in which the dial of the spinner is $\frac{1}{4}$ red and $\frac{3}{4}$, blue, that it is possible (but not very likely) that after 50 spins there are more reds than blues--just the opposite of what we might expect. However, when several committees have completed the activity and we look at the total number of outcomes, we can be fairly certain (though not positive) that there will be more blues than reds. This is one reason why the study of probability is so interesting--we can determine what is expected, but this also means that the unusual is also expected. When children grasp this idea that the unusual can and does happen, they too, become delighted when it occurs. Remember that if we toss 4 coins at one time, we expect 4 heads to occur in the long run about 1 out of 16 tries. Thus, even though 4 heads is unusual, it does occur and we expect it to.

Charts, tables, and other devices are commonly used in an introduction to probability, and those selected here are quite simple. Their use is intended to develop intuition and to help promote systematic ways of proceeding and of recording outcomes obtained. They help children visualize and comprehend some of the basic notions in probability.

Vocabulary: Data.

Materials: Those from Lesson 1 and a Summary Report Chart for each Activity which is made by the teacher.

Suggested Procedure:

Pupil Exercise Sheets from the first lesson can be discussed, if this has not already been done. Pupils can share the results from the activities which they completed at home. After this, show some of the devices which were introduced in the first lesson. State that the boys and girls will now use these to see what outcomes actually do occur.

Assign pupils to committees. Have them look at the Activity Report Sheets while you briefly explain their use, discuss standards for committee work, and make physical arrangements for the pupils to work. You can best determine how to proceed with your group of pupils. Some teachers prefer a few large committees, others several small ones.

Each committee returns its completed report sheet to you. Be sure that the number of the activity is written on this sheet. Another activity might then be assigned, so that more data can be obtained. Let pupils know that their data will be pooled with those obtained by other committees to determine whether patterns can be found and conclusions reached. If materials are available, several identical activities can be performed at the same time by different committees.

As the sheets are handed in, you should enter the results on charts, using a separate chart for each activity. These charts need not be elaborate but should be sturdy enough to use in Lessons 3, 4, and 5. Some teachers have found charts made with a crayon or a felt-point pen on newsprint, construction paper or white wrapping paper adequate. Others have made transparencies for use on the overhead projector. For example:

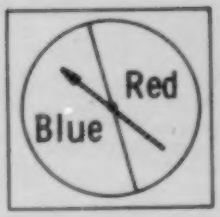
SUMMARY REPORT CHART

Activity 1: Spinner with dial half red and half blue.

Teams	Number of Red	Number of Blue	Total Number of Spins
Team _____			
Team _____			
Team			
TOTALS			

Lesson 2.

Activity 1: Spinning a spinner with dial $\frac{1}{2}$ blue and $\frac{1}{2}$ red.



Directions:

The pointer of the spinner is to be spun and a record made of whether it stops on red or blue. (If it stops exactly on a line between red and blue, make no record, but spin again.)

One member of the committee will serve as recorder. Here is a sample record of 20 spins to show how the count should be kept:

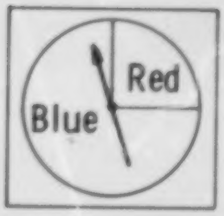
Number of Red		Number of Blue		Total Number of Spins
𝍸 |||	8	𝍸 𝍸 ||	12	20

The other members of the committee will take turns spinning the pointer until a total of 50 spins have been recorded as in the form above. In this class and in math and science classes in higher grades, you will need to keep a record of results from experiments. Practice now by organizing your results and recording them neatly.

Make a table on a full sheet of paper similar to the sample on this page. Head it "Activity 1". Record the results of 50 spins. The person who records the spins and the persons doing the spinning should sign the sheet. Then give it to your teacher.

Lesson 2.

Activity 2: Spinning a spinner with dial $\frac{1}{4}$ red and $\frac{3}{4}$ blue.



Directions:

The pointer of the spinner is to be spun and a record made of whether it stops on red or blue. (If it stops exactly on a line between red and blue, make no record, but spin again.)

One member of the committee will serve as a recorder. Here is a sample record of 20 spins to show how the count should be kept:

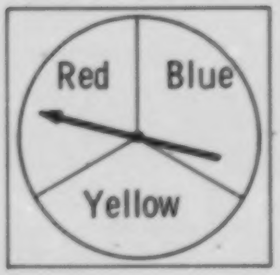
Number of Red		Number of Blue		Total Number of Spins
𝍸 |||	8	𝍸 𝍸 ||	12	20

The other members of the committee will take turns spinning the pointer, until a total of 50 spins have been recorded on another sheet as in the form above. Be sure to put Activity 2 for its heading.

When all 50 spins have been recorded, each committee member should sign the report. Give it to your teacher.

Lesson 2.

Activity 3: Spinning a spinner with dial $\frac{1}{3}$ red, $\frac{1}{3}$ blue, and $\frac{1}{3}$ yellow.



Directions:

The pointer of the spinner is to be spun and a record made of whether it stops on red, blue, or yellow. (If it stops exactly on a line separating two colors, make no record, but spin again.)

One member of the committee will serve as recorder. Here is a sample record of 20 spins to show how the count should be kept:

Number of Red		Number of Blue		Number of Yellow		Total Number of Spins
𝍸 |	6	𝍸 ||||	9	𝍸	5	20

The other members of the committee will take turns spinning the pointer, until a total of 50 spins have been recorded on another sheet as in the form above, headed "Activity 3".

When all 50 spins have been recorded, each committee member should sign the report. Give it to your teacher.

Lesson 2.

Activity 4: Tossing a coin.

heads

tails

Directions:

The coin is to be tossed and a record made of whether it falls with heads or with tails showing.

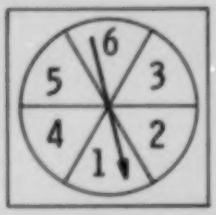
One member of the committee will serve as recorder. Here is a sample record of 20 tosses to show how the count should be kept:

Number of Heads		Number of Tails		Total Number of Tosses
𝍸 ||||	9	𝍸 𝍸 |	11	20

The other members of the committee will take turns tossing the coin, until a total of 50 tosses have been recorded on another sheet as in the form above. Head the sheet "Activity 4".

When all 50 tosses have been recorded, each committee member should sign the report. Give it to your teacher.

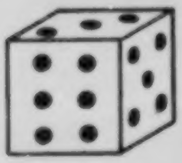
Lesson 2.



Activity 5: Spinning this spinner

or

Tossing a die and counting the number of dots on the top face.



Directions:

The pointer of the spinner is to be spun or the die is tossed and a record made of the results.

One member of the committee will serve as the recorder. Here is a sample of 20 turns to show how the count should be kept:

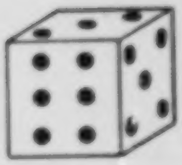
No. of 1's	No. of 2's	No. of 3's	No. of 4's	No. of 5's	No. of 6's	Total Number
////	卌	/	//	///	卌	
4	5	1	2	3	5	20

The other members of the committee will take turns, until a total of 50 spins or tosses have been recorded on another sheet as in the form above, headed "Activity 5".

When 50 results have been recorded, each member of the committee should sign the report. Give it to your teacher.

Lesson 2.

Activity 6: Tossing a die and noting whether the number of dots on the top face is even or odd.



Directions:

The die is to be tossed and a record made of whether the number of dots on the top face is an even number or an odd number.

One member of the committee will serve as a recorder. Here is a sample record of 20 tosses to show how the count should be kept:

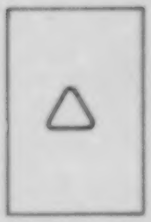
No. of times even		No. of times odd		Total Number of Tosses
𝍸 ||	7	𝍸 𝍸 |||	13	20

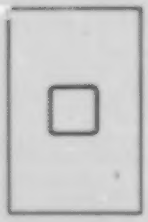
The other members of the committee will take turns tossing the die, until a total of 50 tosses have been recorded on another sheet as in the form above, headed "Activity 6".

When all 50 tosses have been recorded, each member of the committee should sign the report. Give it to your teacher.

Lesson 2.

Activity 7: Choosing one of the two cards pictured here.





Directions:

The two cards are to be placed face down on the desk. (Be sure that the person who is to choose does not know which card is which.) One card is chosen and a record is made of the picture on it.

One member of the committee will serve as recorder. Here is a sample of 20 choices to show how the count should be kept.

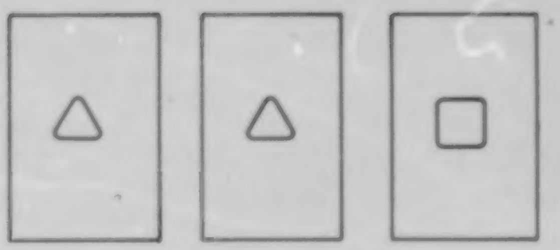
Number of △'s		Number of □'s		Total Number of Choices
卌 ||||	9	卌 卌 |	11	20

The other members of the committee will take turns choosing cards until a total of 50 choices have been recorded on another sheet as in the form above, headed "Activity 7".

When all 50 choices have been recorded, each member of the committee should sign the report. Give it to your teacher.

Lesson 2.

Activity 8: Choosing one of the three cards pictured here.



Directions:

The three cards are to be placed face down on the desk. (Be sure that the person who is to choose does not know which card is which.) One card is chosen and a record is made of the picture on that card.

One member of the committee will serve as recorder. Here is a sample record of 20 choices to show how the count should be kept:

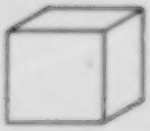
Number of △'s		Number of □'s		Total Number of Choices
𝍸 𝍸 |||	13	𝍸 ||	7	20

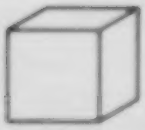
The other members of the committee will take turns choosing cards, until a total of 50 choices have been recorded on another sheet as in the form above, headed "Activity 8".

When all 50 choices have been recorded, each member of the committee should sign the report. Give it to your teacher.

Lesson 2.

Activity 9: Choosing a cube from a box containing one yellow and one red.


red


yellow

Directions:

From a box containing one red cube and one yellow cube, neither visible, a single cube is chosen and its color recorded.

One member of the committee will serve as recorder. Here is a sample record of 20 choices to show how the count should be kept:

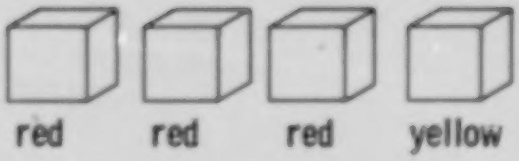
Number of Red		Number of Yellow		Total Number of Choices
𝍸 𝍸 |	11	𝍸 ||||	9	20

The other members of the committee will take turns choosing cubes, until a total of 50 choices have been recorded on another sheet as in the form above, headed "Activity 9".

When all 50 choices have been recorded, each member of the committee should sign the report. Give it to your teacher.

Lesson 2.

Activity 10: Choosing a cube from a box containing one yellow and three reds.



Directions:

From a box containing three red cubes and one yellow cube, none visible, a single cube is chosen and its color recorded.

One member of the committee will serve as recorder. Here is a sample record of 20 choices to show how the count should be kept:

Number of Red		Number of Yellow		Total Number of Choices
𝍸 𝍸 𝍸 |	16	||||	4	20

The other members of the committee will take turns choosing cubes, until a total of 50 choices have been recorded on another sheet as in the form above, headed "Activity 10".

When all 50 choices have been recorded, each member of the committee should sign this report. Give it to your teacher.

LESSON 3

Thinking About What Happened

Introduction.

This lesson is designed to help pupils clarify their concepts of chance and uncertainty. The pupil activities in Lesson 2 are used to introduce a systematic way of thinking about the results of chance events. Pupils are led to discover possible patterns among chance events and to use these patterns for estimating or predicting future outcomes. Finally, pupils are asked to compare the relative likelihoods of different chance events.

Discussion should be informal with pupils being encouraged to state ideas in their own words. Clarity of meaning rather than the use of technical language should be the goal. Teaching procedures are intended as examples rather than prescriptions and should be modified as necessary to meet class needs. You will need to determine the pacing and points of emphasis.

Vocabulary: Reasonable, ratio, estimate, predict.

Materials: Those from Lesson 1 and the Summary Report Charts from Lesson 2.

Suggested Procedure:

Ask the pupils if they made any new discoveries about chance in their committee activity. Encourage discussion and ask pupils who changed their thinking, why and at what point they did so.

Display the Summary Report Charts for Activity 1 - the spinner activity (half red, half blue); Activity 4 - the coin-tossing activity; Activity 6 - the die activity in which the top face is noted as an even or an odd number; Activity 7 - the card-choosing activity with two cards; and Activity 9 - the cube-choosing activity with two cubes.

Ask pupils how these activities are alike. (In each there are two equally likely outcomes.) Raise questions such as: Do you think we'd get exactly the same results if we repeated these activities? (No) Is it possible that the pointer on the spinner would stop on blue every time? (Yes, but it is not very likely.) Is it likely that the coin will be heads 50 times in 50 tosses? (No, but it is possible.) Develop that in these activities we expect each of the two possible outcomes to occur about half of the time. Discuss the charts of Activities 1, 4, 6, 7, and 9 and see if each of the two outcomes did happen about half the time.

Show the Summary Report Chart for Activity 2 - the spinner with dial of one-fourth red and three-fourths blue - and ask questions such as:

> Do these results agree with your expectations? Were there more blues than reds?
>
> How were the reds and blues divided when the spinner was one-fourth red and three-fourths blue? (The division is most likely to be about one red to three blues. The important point is that the ratio of red to blue is most likely to be about one to three.)
>
> Are you more likely to get red on the first spinner or the second spinner? (First.) On which spinner is the chance of getting red the same as the chance of getting blue? (First.)
>
> Is it possible to get only red on the second spinner? (Yes.) Is this likely to happen? (No.)
>
> On which spinner would you be least likely to get 25 reds in 50 spins? (Second one.)

Look at the Summary Report Chart for Activity 10 - a cube-choosing activity with four cubes. Discuss the results and compare them with those from Activity 2 above. Ask questions to help children see how these two activities are alike and to see if the results were about what they expected or predicted.

Show two cubes, one red and one yellow, and place them in a bag.

> If I put one more red cube in the bag and then pick one without looking, is it more likely to be red or more likely to be yellow? (Red, since there are now two reds and only one yellow.)

If the bag contains three red cubes and one yellow cube, how many cubes must I remove to be sure I have removed at least one red one? (Two. Since there is only one yellow cube, any two cubes must include one red one.)

How many must I remove to be sure I have removed at least one yellow one? (Four. Since there is only one yellow cube, all four must be removed to be certain of getting the yellow one.)

When the bag contains three red cubes and one yellow cube, and I pick one cube without looking, how many chances do I have of picking a red one? (Three.) A yellow one? (One.)

Then if I pick one cube, would you say the chances are three out of four that it will be red? (Yes.)

Does this mean I will get a red one? (No. It means I'm three times as likely to get a red as a yellow but it does not mean I am sure to get a red.)

Show the die and say:

Our committees did two activities with a die. How many faces, or surfaces, does it have? (Six.) How many of the faces have an odd number of dots? (Three.) An even number of dots? (Three.) How many have more than two dots? (Four.)

If I toss this die one time, am I more likely to get an odd number or an even number? (The two are equally likely.) Am I more likely to get a number larger than 3 or a number smaller than 3? (Larger.) Am I more likely to get a 1 than a 6? (No. They are equally likely.) Am I more likely to get a 2 than an odd number? (No, less likely, since there are three odd numbers but only one 2.)

If I toss the die six times, is it likely that I will get the same number each time? (No.)

Each of the six numbers has an equal chance each time I toss the die. Does this mean that if I toss it six times I will get each number once? (No.)

If I toss the die 1000 times, am I likely to get each number about the same number of times? (Yes.) Is it possible that I might get no 6's ? (Yes, it is possible but very unlikely.)

How many times must I toss the die to be certain of getting at least one 6? (We cannot tell. I might get a 6 on the first toss but it is possible that I would not get a 6 in a million tosses. The greater the number of tosses, the greater the chance of getting any one of the six numbers.)

Show the Summary Report Chart for Activity 5 and discuss it with the class. Then show Summary Report Chart for Activity 3 - the three-colored spinner - and ask questions as you have done before. Here there are three possible outcomes, each of which is equally likely.

Finally, look at the Summary Report Chart for Activity 8 - the card-choosing activity from three cards - and compare the results with what might be expected.

Help pupils to summarize in their own words what they have learned. This might include the following ideas:

1. When chance is involved, we can never be certain beforehand of the exact outcomes.
2. There are often patterns in large groups of chance events. These patterns help us to estimate more accurately what the outcomes might be if the events are repeated.
3. Sometimes two events are equally likely; sometimes one of them is more likely than the other.
4. We can use activities to check our estimates or predictions about chance events.

Pupil pages 17-20: Pupils' answers should be discussed in class.

Pupil pages 21-23: These optional pages give added opportunity to think about probability. The devices can be constructed quite easily from the directions given in the Appendix. The idea that, in the long run, the results are about what we expect is developed. You would not expect, for example in part c of the first exercise, to get exactly 300 reds in 900 spins. This would be quite unusual, but you do expect to get about 300 reds. (We can determine by mathematics, that if we were to spin this spinner for many sets of 900 spins, we would find that the number of reds would be between 270 and 330 in over 95 per cent of the sets of 900 spins.) Exercise 2(c) is of the same nature. (Here the number of reds would be between 222 and 278 in over 95 per cent of the sets of 1000 tosses.)

On page 23 we would expect each face to be up about 30 times in 180 tosses. (Again, by mathematics we can determine that about two-thirds of the time in sets of 180 tosses, the face with 2, or any other face, will be up between 25 and 35 times. In over 95 per cent of the sets of 180 tosses, we can expect the 2 to appear between 20 and 40 times.)

Exercises - Lesson 3.

Here are ten statements about chance events. If you think a statement is true, put a T in the blank after the statement. If you think the statement is not true, put an F in the blank.

1. If a tossed coin does not stand on its edge, it is certain to be either heads or tails. T
2. If we toss a coin once, we are as likely to get a head as a tail. T
3. If we toss a coin 100 times, it may be heads 0 times or 100 times or anything in between. T
4. If we toss a coin 1000 times, it is very unlikely that we will get 900 tails. T
5. Whether we get heads or tails when we toss a coin is a matter of chance. T
6. You might toss a coin 1000 times without getting a single head. T
7. If a box contains two blue marbles and one red one, and you pick one marble without looking, the chances are 2 out of 3 that it will be blue. T
8. In Exercise 7, you have one chance in three of picking a red marble. T

9. In Exercise 7, your chances of picking a green marble are zero. T

10. Joe is eight years old. It is more likely that he is four feet tall than ten feet tall. T

Read the following statement carefully and then answer Questions 11 to 15.

> A spinner has a dial which is one-fourth white and three-fourths red.

11. If you spin the pointer 10 times, are you likely to get the same number of reds as whites? no

12. Are you likely to get more whites than reds? no

13. If the chances of getting red are 3 out of 4, what are the chances of getting white? 1 out of 4

14. Can you be certain of getting at least one red in 10 spins? no

15. Is it very likely that you will get no reds in 10 spins? no

Read the following statement carefully and then answer Questions 16 through 20.

James has three green marbles and
two blue marbles in his pocket.

16. How many marbles must he remove to be sure of getting a blue marble? 4

17. How many marbles must he remove to be sure of getting both the blue ones? 5

18. How many marbles must be removed to be sure of getting both colors? 4

19. How many marbles must be removed to be sure of getting a green one? 3

20. If James removes one marble, there are three chances out of ___?___ that it will be a green one. 5

Think about some things that are certain to happen. Think about some things that might happen, and about other things that just can't happen. Then answer questions 21, 22, and 23 .

21. List three things that you know are certain to happen.

a. ____

____

b. ____

____

c. ____

____

22. List three things that may or may not happen.

a. ____

____

b. ____

____

c. ____

____

23. List three things that cannot happen.

a. ____

____

b. ____

____

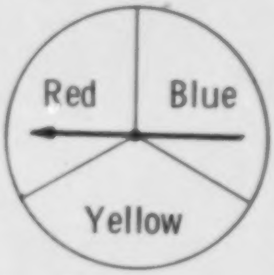
c. ____

____

Things To Do At Home - Lesson 3.

1. Make a spinner with a dial $\frac{1}{3}$ red, $\frac{1}{3}$ blue, and $\frac{1}{3}$ yellow. Use your imagination. Spinners can be made from paper plates, plastic tops from a paper cup, cardboard, ice cream sticks, and so on. Place a bead under the pointer, or anything to keep it up off of the dial. Stick a pin through the pointer, the exact center of the dial, and into something like an eraser.

   Spin the pointer 100 times. Keep track of the outcomes in a table such as this:



	Blue	Red	Yellow
Tally			
Total			

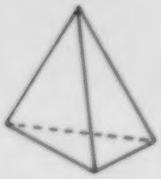
   a. How many times did the pointer stop on blue? ________

   On red? ________ On yellow? ________

   b. Is each of these colors equally likely? Yes

   c. If you spin the pointer 900 times, you expect it to stop on red about [✓ 300] 400 500 times. (Circle your answer.)

2. Use the directions in the Appendix to construct a tetrahedron. Color one face red, one blue, another yellow, and the last green.



Toss the tetrahedron 100 times and note the face that is down. Keep track of the outcomes in a table such as this:

	Red	Blue	Green	Yellow
Tally				
Total				

a. How many times did the tetrahedron fall on Red ? ________

Blue? ________ Green? ________ Yellow? ________

b. Add the number of times it fell on red and on blue. ________

Add the number of times it fell on green and on yellow. ________

Is each of these sums about $\frac{1}{2}$ of the total number of tosses, or about $\frac{1}{4}$ of the total number of tosses? About ________.

c. If you throw the tetrahedron 1000 times, about how many times do you expect it to fall on red? 350 On either blue or red? 500

3. Make up a game that two people can play so that each person will have an equal chance of winning. Explain it to the class. Describe any material needed, such as spinners or dice. Be sure your rules are clearly stated.

4. Toss a die or a cube which has its faces numbered from 1 through 6. The Appendix has directions for constructing a cube (hexahedron), or you may use a die from some game.

Toss it 180 times. Use a table to keep a record of the results.

	Number of 1's	Number of 2's	Number of 3's
Tally			
Total			

	Number of 4's	Number of 5's	Number of 6's
Tally			
Total			

a. Did each face come up at least once? ________

b. The faces are either odd (1, 3, 5) or even (2, 4, 6). On 180 tosses, about how many outcomes would you expect to be even numbers? 90 In your experiment, how many were even numbers? ________

c. On a die or cube, each of the six faces has an equal chance to be up. In 180 tosses, how often would you expect 2 to be up? 30 5 to be up? 30 In you experiment of 180 tosses, how many times was 2 up? ________ 5? ________

d. If you tossed a die 6000 times, about how often would you expect each face to be up? 1,000

LESSON 4

Using Graphs to Learn About Chance

Introduction.

This lesson helps pupils with the techniques for gathering, tabulating, graphing, and interpreting data which they generate by tossing a coin and tossing a die.

If pupils are already efficient in these techniques, the lesson will move quite rapidly and might even be considered optional. You can decide whether your class will profit from it. The ideas in Activities 11 and 12 are not a radical departure from those already presented but serve to help the pupils refine their thinking and sharpen their intuition about chance events by analyzing the results of a large number of trials.

Vocabulary: Tabulate, horizontal, vertical, intersection of lines, scale.

Materials:
(1) A coin and a die for each of the ten teams.
(2) Summary Report Chart of coin-tossing from the previous lesson.

Suggested Procedure:

Tossing a Coin

Display the Summary Report Chart of Activity 4 - a coin-tossing activity. Emphasize that it shows the total number of heads and tails from a few committee experiments. Suggest that it would be possible to compare the results obtained by each of 10 different teams if we could display the information in some easy way. Explain that each of these 10 teams will toss a coin and will later toss a die to get data which will be graphed. This number of teams has been chosen for convenience in later computation, so it is necessary that there be 10 teams. We want a total of 100 tosses of a coin and 600 tosses of a die. Assign children to the 10 teams and give a coin to each team. Briefly explain page 24 and ask pupils to predict what they think the results will be. Letter the sheets from A through J as each team hands one to you.

Record the data on the chalkboard, as on Pupil page 25, and have the children complete page 25 in their books. Then discuss the five questions at the bottom of page 25. Question 4 is concerned with what we expect, in the long run, when we toss a coin. We do expect about half of the tosses to be heads. (In our sample of 100 tosses, we can use mathematics to compute that the number of heads will be between 40 and 60 in about 95 per cent of the sets of 100 tosses.) You might write on the board the fractions $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, and ask which best represents the fraction of heads in the total number of tosses.

Have pupils look at page 26 and explain, as is necessary, the graph form and the manner of graphing each team's data. Note that the symbol " O " is placed at the intersection of a horizontal and a vertical line. Do not connect the "O's" with line segments. Point out the graph form for the total number of heads in 100 tosses and discuss it with the class. Call attention to the fact that the scale on it is different from the graph form to its left and that it represents just 10 times the numbers on the first form.

When the pupils have completed the graphs, emphasize the ease with which certain questions can now be answered. For example: Which team had the largest number of heads in 10 tosses? How many teams reported either 4, 5, or 6 heads in 10 tosses? How many teams reported either 0, 1, or 2 heads in 10 tosses? Did any team report either 9 or 10 heads? Where do the points of the graph of the teams' data seem to cluster - at the top, middle, or bottom of the graph form and how does this compare with the graph of the total data?

The questions at the bottom of page 26 can be answered correctly in several ways. We are trying to develop the idea that in the long run, heads will occur about half of the time and question 1 is concerned with this. Show that although the number of heads reported by the teams is quite varied, the number of heads on 100 tosses is more stable--that is, about half of the tosses are heads. Question 2 might be answered either "yes" or "no". In the discussion bring out, for example, that 57 heads in 100 tosses is closer to $\frac{1}{2}$ than is 6 heads in 10 tosses, even though 57 is 7 away from 50 while 6 is only 1 away from 5, and we expect heads to occur in about one-half of the tosses. Question 3 is based on this same idea of expecting about one-half of the tosses to be heads in the long run. Explain as is necessary, depending upon the background of your children.

Tossing a Die

> We have tabulated and graphed the results of tossing a coin. Now we'll work with a die. How many faces does a cube have? Is a 1 as likely to be up on one toss as a 6 ?

You might use the same ten teams as before and give each one a die. Briefly explain Activity 12 on page 27 and ask pupils what they expect the results will be. Letter their sheets from A through J as each team hands its sheet to you and record the data on the chalkboard as on Pupil page 28. Each pupil then completes page 28 in his text. Discuss pupils' answers to the six questions at the bottom of page 28 and make any appropriate comments concerning the results obtained by the various teams. In Question 6 we expect each face to be up about 100 times. In over 95 per cent of the sets of 600 tosses, a given face will be up between 81 and 119 times.

On Pupil page 29 explain, if necessary, how to draw the graph. Pupils may need to be informed of the letter of their team so that they know which data to use. Discuss their responses to the two questions at the bottom of this page. We would expect each face to occur about 10 times in 60 tosses of a die but because 60 is not a large number of tosses, some pupils may find a face occurring only 4 or less times or perhaps as many as 16 times or more.

Pupils can draw the graph on page 30 and should not connect the "X's" with line segments. Then ask such questions as: On all our 600 tosses, did any face fail to appear? What is the largest number of times that any face appeared? The smallest? Did any face appear on more than half of the tosses? Would you expect it to? If, after many tosses, we found that a 6 had not occurred, would we expect a 6 on the next toss?

The two questions at the bottom of page 30 will require discussion. Here again, the idea is that the total class results should be "more like" the results we would expect than would be the case from a smaller number of tosses. For example, we expect about $\frac{1}{6}$ of the tosses to be a 3, so in 600 tosses we expect 3 to appear about 100 times. If it appeared only 80 times, it still is closer to the $\frac{1}{6}$ of the times we expect than is its occurrence 7 times in 60 tosses. You might suggest that pupils draw a horizontal line on the total graph form at the place which represents the number of expected occurrences of each face (at 100) and see how the results of 600 tosses cluster around this line. Pupils could then draw, in the same way, a horizontal line (at 10) on the graph form on page 29 and again see how their results cluster around this line.

Lesson 4.

Activity 11: Tossing a coin -- a team activity.

Directions:

The coin is to be tossed ten times and a record made of whether it falls heads or tails.

One member will toss the coin and another member will record the result. Here is a sample record of 10 tosses to show how to record the results.

Sample Record

Number of Heads		Number of Tails		Total Number of Tosses
////	4	𝍸 /	6	10

On another sheet make a table similar to the one above. Keep track of your results for 10 tosses. Each member of the team should sign the sheet. Your teacher will ask for it soon, because your results will be used in a class activity.

25

Lesson 4.

Class Data Sheet for Tossing a Coin -- To accompany Activity 11.

Team	No. of Heads in 10 Tosses	No. of Tails in 10 Tosses
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
Total		

1. Which team had the largest number of heads in 10 tosses? _______
   the largest number of tails in 10 tosses? _______
2. About how often would you expect heads in 10 tosses? 5
   in 100 tosses? 50
3. Each team tossed the coin 10 times. In all, how many times was the coin tossed? 100 What is one-half of this number? 50
4. Out of all these tosses, how many times did heads occur? _______
   Is this about one-half the number of tosses? _______
5. Would we need to record the number of tails on our Class Data Sheet? no

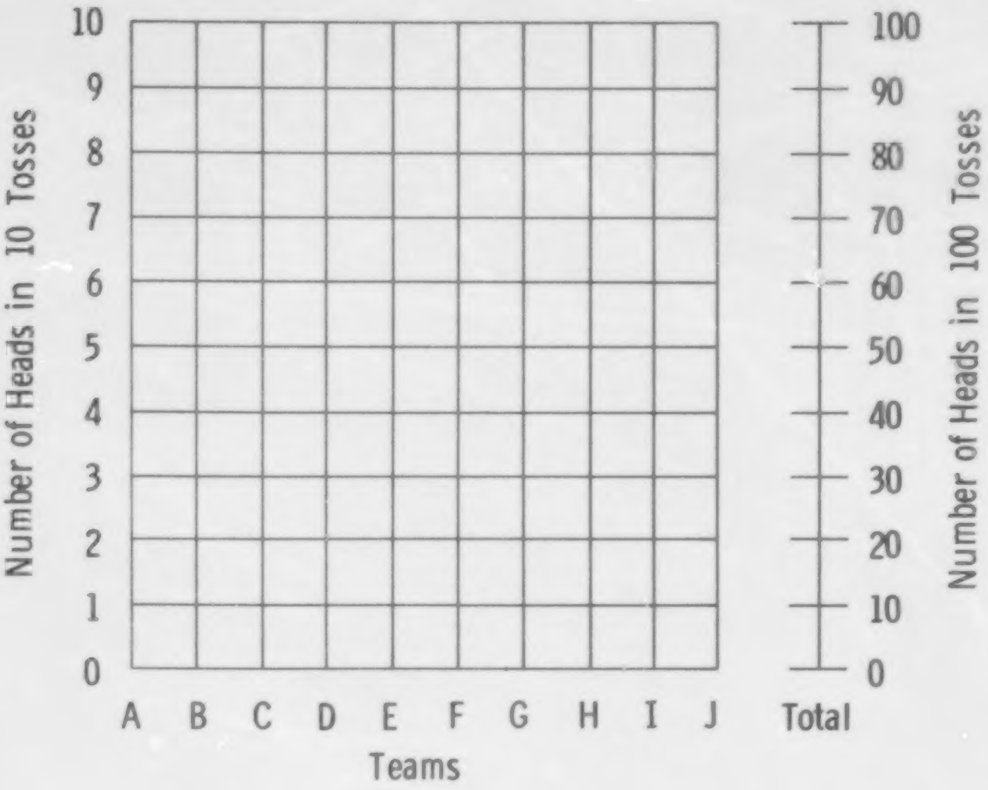
Lesson 4.

Graph Sheet for Tossing a Coin -- To accompany Activity 11.

Directions:

Graph each team's data with an "O". (Use the scale on the left.)

Graph the class total data with an "X". (Use the scale on the right.)



Compare the graph of each team's results of 10 tosses with the graph of the total data of 100 tosses.

1. In general, which is more nearly what you would expect, the various teams' results or the total results? _______
2. Is the number of heads in 100 tosses more nearly what you would expect than the results *your* team had in 10 tosses? _______
3. Is a larger number of tosses more likely to result in what you expect than just a few tosses? Yes

Lesson 4.

Activity 12: Tossing a die -- a team activity.

Directions:

The die is to be tossed sixty times and a record made of the number of dots on the top face.

One member will toss the die and another will record the result. Here is a sample record of 20 tosses.

Sample Record

	No. of ⚀	No. of ⚁	No. of ⚂	No. of ⚃	No. of ⚄	No. of ⚅
Tally	//	////	///	//	///	𝍸 I
Total	2	4	3	2	3	6

Use a sheet of paper to make a table similar to the one above. Toss the die 60 times and record the results in your table. Each member of the team should sign the sheet. Your teacher will soon ask for it because your results will be used in a class activity.

Lesson 4.

Class Data Sheet for Tossing a Die -- To accompany Activity 12.

Team	No. of ⚀	No. of ⚁	No. of ⚂	No. of ⚃	No. of ⚄	No. of ⚅	Total No. of Tosses
A							
B							
C							
D							
E							
F							
G							
H							
I							
J							
Total							600

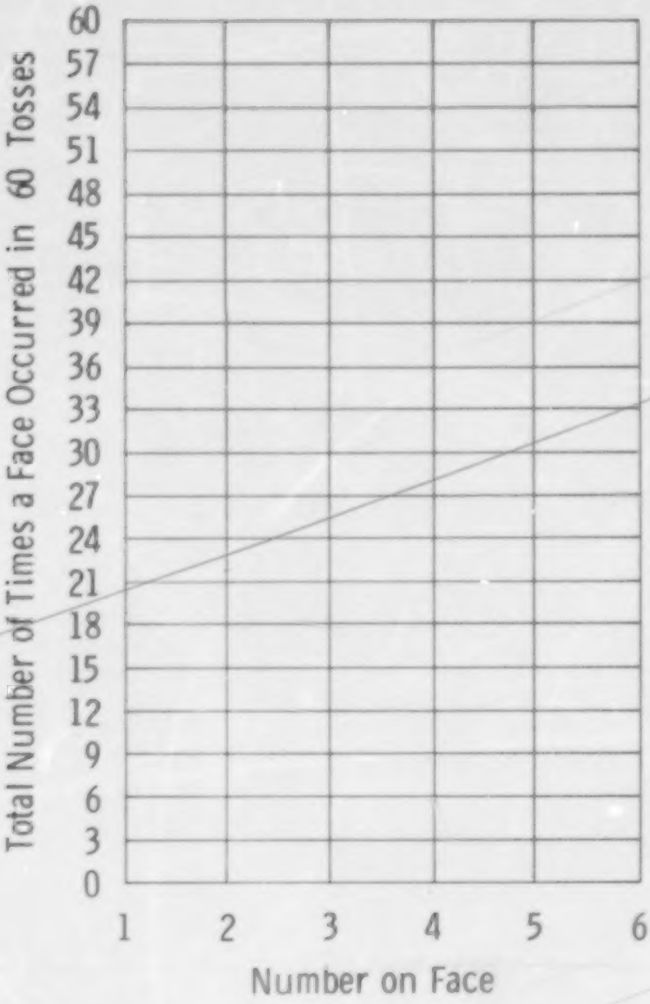
1. From the totals, which face of the die was up the most? ________
2. Are any two or more totals the same? ________ Which? ________
3. Did any team fail to get all six numbers? ________
4. Would you expect that on 60 tosses, each number would come up at least once? Yes
5. If we tossed a die 1000 times, could we be sure that every number would come up at least once? no
6. In 600 tosses of a die, about how many times would you expect each face to be up? 100

Lesson 4.

Graph Sheet for Tossing a Die -- To accompany Activity 12.

Directions:

Graph your team's data with an "O".



1. Did you have any results which you think are unusual?

____

2. What makes you think these results are unusual?

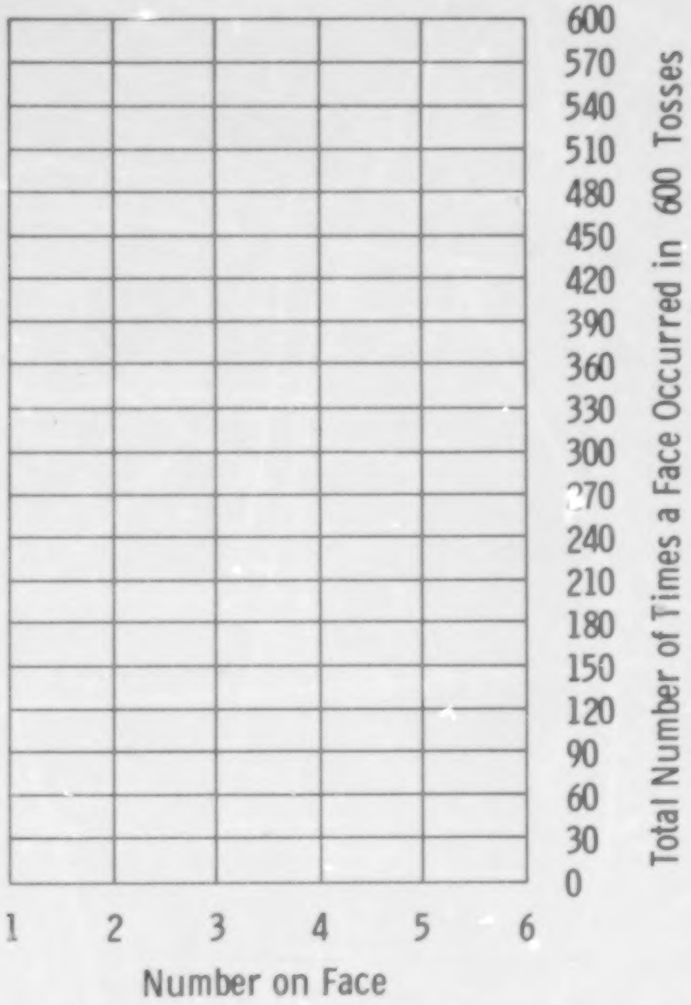
____

Lesson 4.

Graph Sheet of Total Class Results for Tossing a Die -- To accompany Activity 12.

Directions:

Graph the total class results with an "X".



This graph form is the same form that you used to graph your team's results. However, the numbers on the right side are ten times those on your team's graph form. This is a graph of 600 tosses instead of 60 tosses. Compare your team's graph with this graph.

1. Does your team's graph seem to be farther from the expected results, or are the total class results farther from the expected results? __________
2. Can you explain why this is so? _________

LESSON 5

Using Bar Graphs With Probability Experiments

Introduction.

This lesson uses bar graphs to reinforce ideas relating to probability. Pupils construct and interpret bar graphs using results of the Activities in Lesson 2. Select for discussion and graphing the data from one team for each activity. (You may also have pupils draw a graph of the combined data from those Activities which were carried on by more than one team.)

Many interesting observations can be made from the graph on Pupil page 31 which will show the results from all ten Activities. For example, Activities 1, 4, 6, 7, and 9 involve two equally likely outcomes, and we would expect the results to cluster about the 25 on the graph. Activities 2 and 10 involve outcomes which are not equally likely, and we would expect a different picture. Activities 3 and 8 should show a third pattern and Activity 5 another pattern. Picturing all this information on one graph form should be quite revealing but time must be taken to discuss each graph and how and why it differs from the others.

Vocabulary: Bar graph.

Materials:
(1) Summary Report Charts of results from Activities 1 through 10 from Lesson 2.
(2) A table on the chalkboard or on newsprint, or for use on the overhead projector, as indicated in the next paragraph.

Suggested Procedure:

> When we worked in teams, we recorded the results of ten different activities. Here are the results that some teams found: (Use actual results as reported and have a table on the chalkboard or on a large sheet of newsprint or for the overhead projector. See example on the next page.)

	Activity	Number of trials	Result
Act. 1	Spinning a spinner ($\frac{1}{2}$ red, $\frac{1}{2}$ blue.)	50 spins	22 blues
Act. 4	Tossing a coin (Heads or tails.)	50 tosses	29 heads
Act. 6	Tossing a die (Odd or even.)	50 tosses	30 even
Act. 7	Choosing a card (Triangle or square.)	50 choices	19 triangles
Act. 9	Choosing a cube (Red or yellow.)	50 choices	20 reds

Ask pupils in what way these activities are alike and if they notice anything similar about the results.

Teams also reported these results: (Add these to the table on the board or chart and again use actual data.)

Act. 2	Spinning a spinner ($\frac{1}{4}$ red, $\frac{3}{4}$ blue.)	50 spins	32 blue
Act. 10	Choosing a cube (3 red, 1 yellow.)	50 choices	35 reds

Have pupils tell in what way a result of blue on a spinner with a dial $\frac{1}{4}$ red and $\frac{3}{4}$ blue is like a result of red in choosing a cube from a box containing three red cubes and one yellow cube. Bring out that there are three out of four chances of getting a red cube from the box.

Other outcomes reported by two committees were: (Add actual data to the table, such as:)

Act. 3	Spinning a spinner ($\frac{1}{3}$ red, $\frac{1}{3}$ blue, $\frac{1}{3}$ yellow.)	50 spins	14 reds
Act. 8	Choosing a card (2 triangles, 1 square.)	50 choices	20 squares

Ask how these two activities are alike and help pupils determine that the chance is one out of three for red on the spinner and there is one chance in three of choosing the card with the square.

A committee reported these results for Activity 5:
(Use actual data to add to the table.)

Act. 5 Tossing a die (Number of dots up.)	50 tosses	7 ones

Discuss the fact that each face on a die is equally likely to be up when a die is tossed, that a given face has just 1 chance in 6 to be up.

Let's show all the results on one graph and see
how it looks. Maybe we can find some patterns.

Pupils can turn to page 31, "Graph of Committee Activities" and complete their graphs as you graph on the board. This bar graph may be new to some pupils. It can be related to the graphs they have previously made by reteaching, as necessary, how to plot the points for different numbers between 0 and 50, using the graph you have drawn on the board or on large newsprint.

Look at the completed graphs and lead pupils to see that Activities 1, 4, 6, 7, and 9 are concerned with two equally likely events and that the graphs picture the frequency of one of them. We would expect this frequency to be about half the number of trials.

In Activities 2 and 10, the outcomes are not equally likely but favor blue for the spinner and red for the cubes. The chances are three out of four that blue will occur on the spinner and three out of four that a red cube will be chosen. It might be well here to ask what would be the picture if we'd graphed the number of reds on the spinner and the number of yellow cubes chosen. How would the graph look then? This is really laying a foundation for complementary events, which will be discussed later. In Activities 3 and 8 chances are one out of three of getting red on the spinner and one out of three of choosing a card with a square. In Activity 5, the chances are one out of six that a 1 will be up when the die is tossed.

Then compare the graphs of Activities 1, 4, 6, 7, and 9 with Activities 2 and 10 and with Activities 3 and 8. Lead pupils to see that the data from the first five tend to cluster about 25, from the next two about 37, and from Activities 3 and 8 about 17. We would expect the data from Activity 5 to be about 8 but because this is from just 50 tosses, it might vary considerably. If the combined data obtained by all teams were graphed, this clustering would be even more evident.

Before leaving bar graphs, you might want to develop other ideas such as the bars could be placed horizontally instead of vertically. It would be desirable to construct a graph which shows a compilation of all team reports. For example, if three teams completed Activity 1 (50 spins of a spinner with a dial of $\frac{1}{2}$ blue and $\frac{1}{2}$ red), you might want to graph the number of reds in 150 spins.

In discussing the graphs, reinforce the ideas that some things are more likely to happen than others and that graphs help us see patterns and note relationships which might not be evident otherwise.

Pupil pages 32-35: These pages give practice in making and interpreting bar graphs. You might want to omit those exercises which are too easy for your class. On page 33, the results listed for Bill are unusual. You might like to have children toss a coin to see if they find heads do occur this many or more times in 50 tosses. We expect that on 50 tosses, the number of heads will be between 43 and 57 over 95 per cent of the time. Exercise f on page 34 may require some explanation so you may need to do this as a class activity. Exercise h on this same page should be discussed because some pupils will say they would rather guess on 10 tosses because then they "couldn't miss it very far." This is another opportunity to develop the idea that in a large number of trials, our results will probably be close to what we expect, for example, when we say, "We expect heads to occur about one-half the time." Thus, 236 heads in 500 tosses is closer to one-half than is 6 heads in 10 tosses. Question h on page 35 is similar to this.

Pupil pages 36 and 37: Exercises h and i on page 37 help children determine what to expect when many trials are performed. We would expect blue on spinner 2 about $\frac{1}{8}$ of the time or about 125 times in 1000 spins. (If many children spin a spinner 1000 times, the number of blues that over 95 per cent would get would be between 104 and 146. It would be very unusual to get as few as 100 blues or more than 150 blues.)

On spinner 3 we would expect the number of reds to be about $\frac{1}{3}$ of the total number of spins or about 333 in 1000 spins.

Further Activity: Here is a game some children might enjoy playing. It is written so that you can duplicate it for them or you may wish to just explain it. Players will want to determine who has the better chance to win and one way to do this is described.

Squares and Products

Find a partner who will play a game with you. You will need two dice of different colors, for example, red and white, and the (3-colored) spinner marked 1 to 6--or you could use 3 dice. You will also need a piece of paper to keep score. One player is "Squares"; the other is "Products". "Squares" spins the spinner or if you do not have the spinner, he can toss one die. His score is the square of the number he gets. (To find the square of a number, multiply the number by itself: The square of 1 is 1 × 1, or 1; the square of 2 is 2 × 2, or 4; the square of 3 is 3 × 3, or 9, and so on.) "Products" tosses the two dice. He multiplies the numbers of dots on the two dice. Their product is his score. If the spinner shows 3 and the dice show 5 and 2, "Squares" gets 3 × 3, or 9, and "Products" gets 5 × 2, or 10. If the spinner shows 4 and the dice show 2 and 6, "Squares" gets 4 × 4, or 16, and "Products" gets 2 × 6, or 12. Their scores might look like this:

	Spinner	Dice	Squares	Products
First round	3	5, 2	9	10
Second round	4	2, 6	16	12

The winner is the player with the higher score after 15 rounds. Do you know which player is more likely to win? You can find out. Add the squares of the numbers 1, 2, 3, 4, 5, and 6. Divide their sum by 6. This is the average possible score for "Squares". Now find all the possible products. A table is helpful in finding these.

Red Die

White Die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8		
3						
4						
5						
6						

Now add all thirty-six products and divide the sum by 36. This is the average score for "Products". Who has the higher average possible score? Who is more likely to win the game?

Things to think about! The average score for the red die is ______. The average score for the white die is ______. The product of these two numbers is ______.

GRAPH OF COMMITTEE ACTIVITIES

Activity	1 Spinner 1/2 Red 1/2 Blue	4 Coin	6 Die	7 2 Cards 1 with △ 1 with □	9 2 Cubes 1 Red 1 Yellow	2 Spinner 1/4 Red 3/4 Blue	10 4 Cubes 3 Red 1 Yellow	3 Spinner 1/3 Red 1/3 Blue 1/3 Yellow	8 3 Cards 2 with △ 1 with □	5 Spinner or Die
Choice	Blues	Heads	Even	Triangles	Reds	Blues	Reds	Reds	Squares	Ones
50										
48										
46										
44										
42										
40										
38										
36										
34										
32										
30										
28										
26										
24										
22										
20										
18										
16										
14										
12										
10										
8										
6										
4										
2										
0										

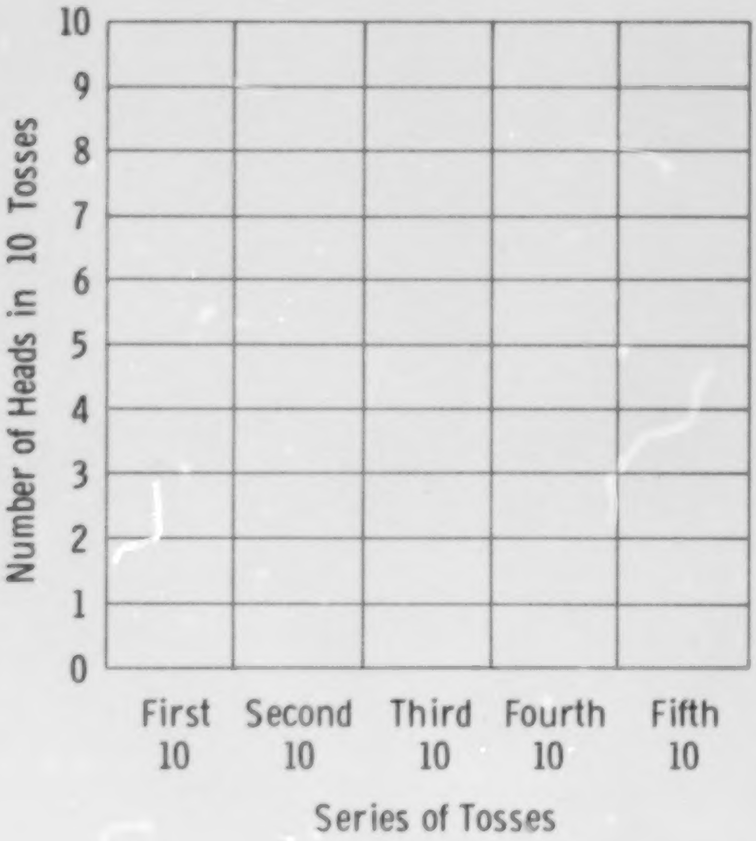
Exercises - Lesson 5.

1. Jim tossed a coin 50 times in groups of 10 tosses. He kept a record of the results by tallies. H stands for heads. T stands for tails.

Tosses

	First 10		Second 10		Third 10		Fourth 10		Fifth 10	
	H	T	H	T	H	T	H	T	H	T
Tallies	𝍸 II	III	IIII	𝍸 I	𝍸 I	IIII	III	𝍸 II	𝍸 I	IIII
Number of heads	7		4		6		3		6	

a. Draw a bar graph to show the number of heads Jim got on each group of 10 tosses.



b. Out of 50 trials, how many heads did he toss? 26

c. On which series did he have the most heads? First

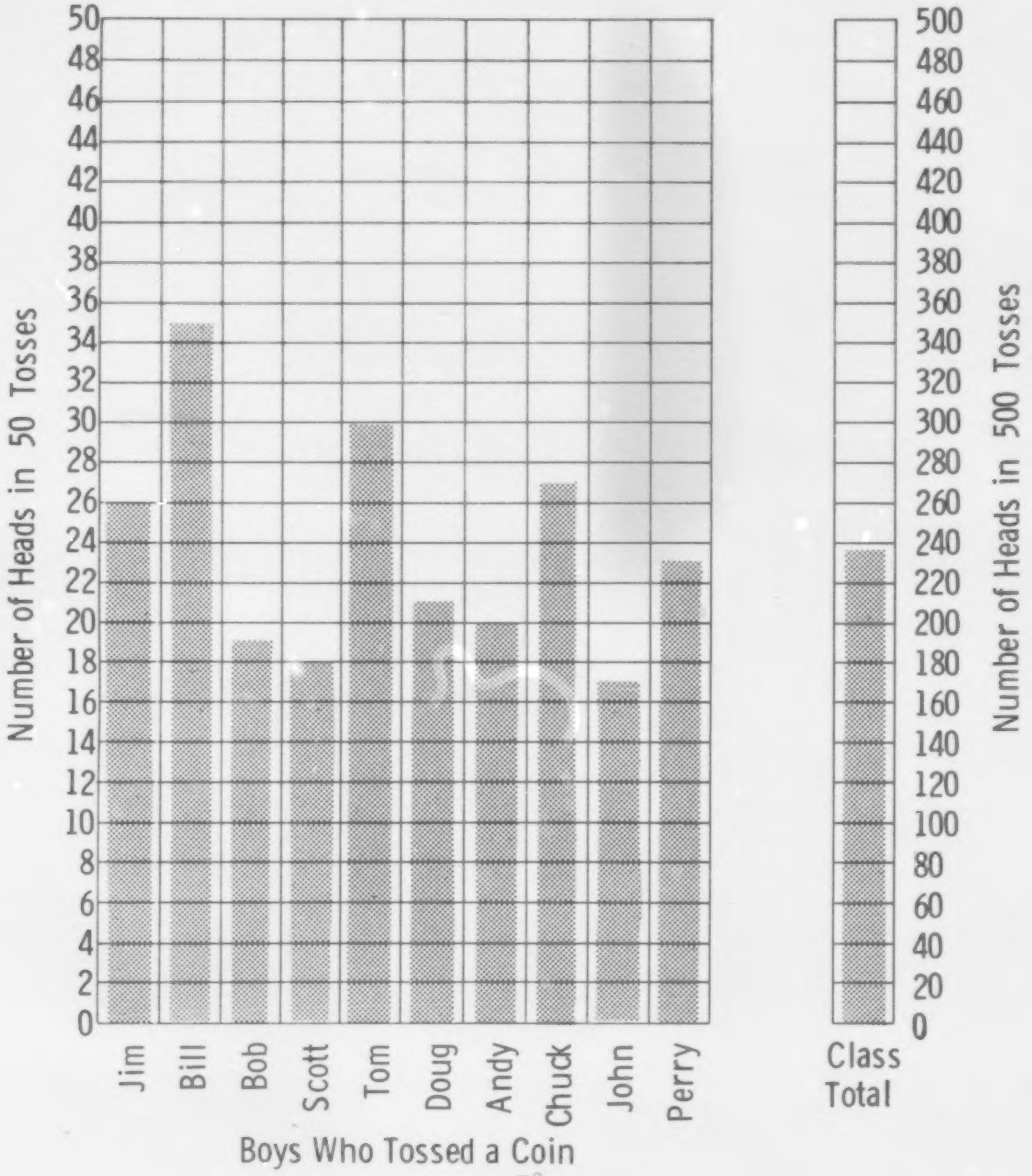
d. On which series did he have the most tails? Fourth

e. Did the number of heads equal the number of tails on any one series? no

2. Nine other boys in Jim's class did the experiment, too. Their results for 50 tosses were:

Bill	35	heads	15	tails
Bob	19	heads	31	tails
Scott	18	heads	32	tails
Tom	30	heads	20	tails
Doug	21	heads	29	tails
Andy	20	heads	30	tails
Chuck	27	heads	23	tails
John	17	heads	33	tails
Perry	23	heads	27	tails

Make a bar graph to show the number of heads these ten boys tossed. Use the scale on the left. Make another graph of the total number of heads in 500 tosses. Use the scale on the right.



2. (Continued)

a. Who are the boys who tossed more heads than tails? Jim

Bill, Tom, Chuck

b. Who tossed the smallest number of heads? John

Who tossed the largest number of tails? John

Is this surprising? no Why? The greater the number of heads on 50 tosses, the smaller the number of tails because the number of heads plus the number of tails must equal 50.

c. Which boy, do you think, might have been most surprised by the results he had in 50 tosses of the coin? Bill

Why? The number of heads is farthest from 25.

d. Each of the ten boys tossed a coin 50 times. This makes a total of 500 tosses. How many of these tosses were heads? 236

Without counting, how many of these tosses were tails? 264

e. How many boys tossed more than 25 heads? 4

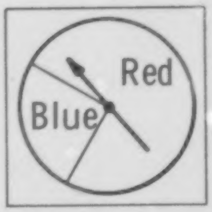
f. Draw a horizontal line across the graph so that about as many of the boys have results above the line as below. At what number does this line intersect the left edge of the graph? 22

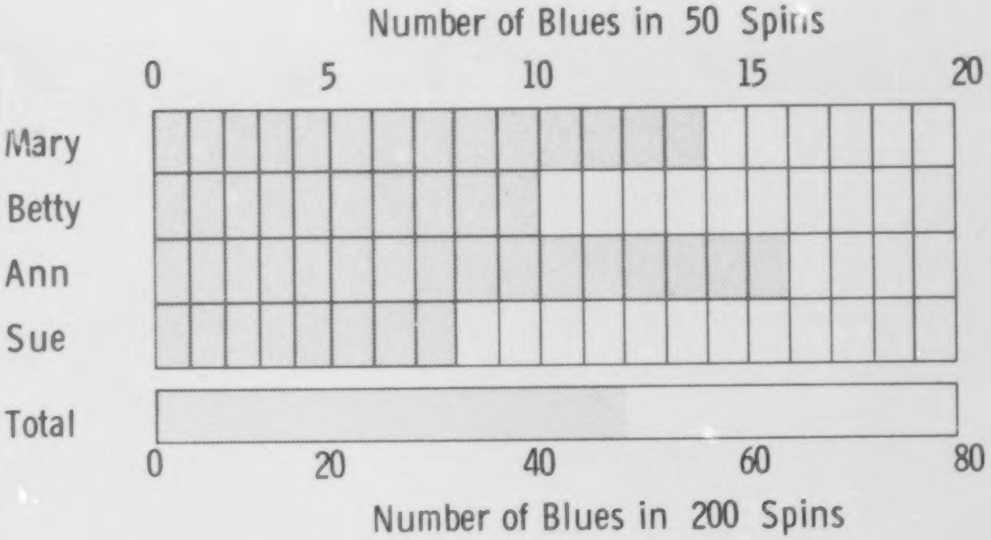
g. At what number does this line intersect the graph of the class total? 220 Is this about the same as the number of heads the boys tossed in 500 tosses? Yes, it is close to it

h. Would you rather tell how many heads you expect in 10 tosses of a coin or in 500 tosses of a coin? ______ tosses.

(See explanation in the T.C.)

3. Below is a bar graph of the results some girls found in using the spinner at the right. You can read it in the same way you do other bar graphs. Look at it carefully, and you will see how to do this. Use it to answer a through h .





a. Who had the smallest number of blues in 50 spins? Sue

b. Who had the largest number of blues in 50 spins? Ann

c. How many reds did Betty get in 50 spins? 40

d. Which of these fractions tells about how much of the dial is blue?

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{3}{4}$ 1/4

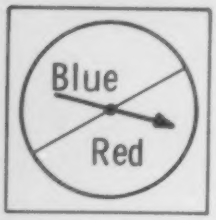
e. Did any girl get 25 or more blues? no

f. How many times in all was the spinner spun by the girls? 200

g. How many of these spins ended on blue? 48 Is this about the number of blues you would expect on 200 spins? yes

h. Would you rather guess the number of blues on 20 spins or on 200 spins? 200 spins. Sue 70

4. Here are some more spinners and graphs.



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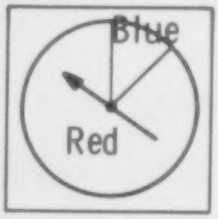

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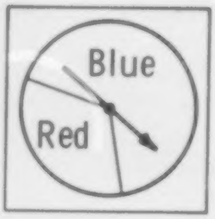

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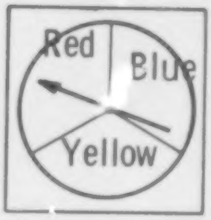
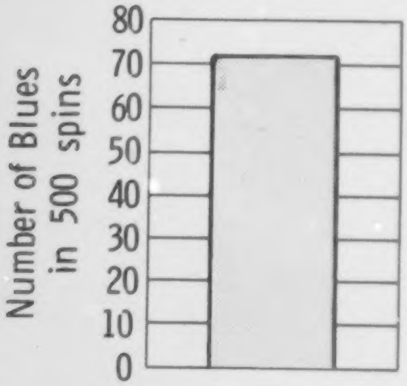
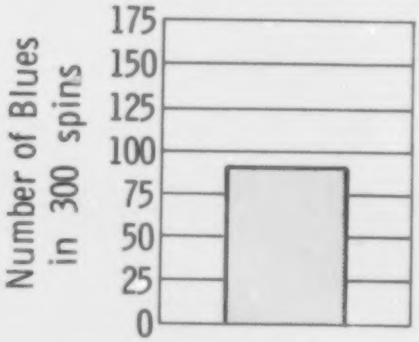

Red
Blue
Yellow

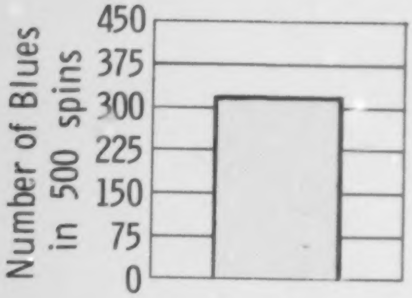
4



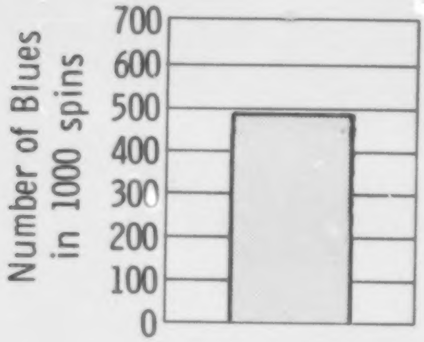
Graph A



Graph B



Graph C



Graph D

a. Graph A was probably made by using data from spinner 2.

b. Graph B was probably made by using data from spinner 4.

4. (Continued)

c. Graph C was probably made by using data from spinner 3.

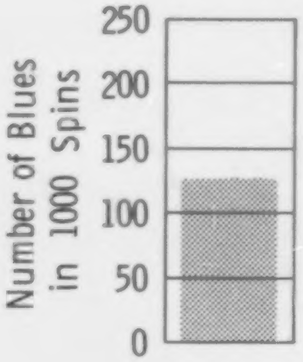
d. Graph D was probably made by using data from spinner 1.

e. Which spinner would you choose if you wanted to be most likely of getting blue? 3

f. On which spinner is red more likely than blue? 2

g. One of the spinners is spun 10,000 times. Blue was the result 3,300 times. Which spinner would you expect was used? 4

h. Spinner 2 is spun 1,000 times. Draw a bar graph to show the number of blues you would expect.



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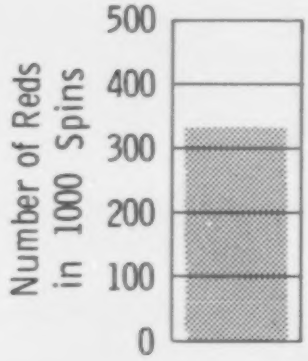
i. Spinner 3 is spun 1,000 times. Draw a bar graph to show the number of reds you would expect.



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LESSON 6

Using Numbers to Express Probability

Introduction.

Up to this point pupils have gathered data from a variety of activities and have learned to summarize it in tables and to represent it visually through the construction of graphs. The probability or chance of various results has been discussed informally on the basis of intuition and hunches. The graphs and charts provide a systematic way of organizing data so that they can be studied and analyzed to discover relationships and patterns. These relationships and patterns enable us to formulate hypotheses and to draw various conclusions.

This lesson begins with a review of the informal ideas pupils have gained from previous lessons. Pupils are then guided to compare the results of different activities; to identify those activities which produce like results and those which produce unlike results; and to discover the causes which may underlie their likenesses and differences. As a natural outgrowth of these comparisons, pupils are introduced to the use of rational numbers as measures of chance and then to the use of number sentences to express probability.

The Suggested Procedure outlines a logical sequence for developing the concepts in this lesson. You may choose to condense or extend the lesson to provide the pacing best suited to your pupils.

Vocabulary: Probability, number sentence, express, mathematician.

Materials:
(1) Charts and graphs from Activities 1-10.
(2) Spinner ($\frac{1}{2}$ red, $\frac{1}{2}$ blue).
(3) Spinner ($\frac{1}{4}$ red, $\frac{3}{4}$ blue).
(4) Spinner ($\frac{1}{3}$ red, $\frac{1}{3}$ blue, $\frac{1}{3}$ yellow).
(5) Colored cubes or beads (1 red, 1 white, 1 blue, 1 yellow).

Suggested Procedure:

Begin by having pupils recall the ideas developed during discussion of experimental activities. (Have charts and graphs available.) Ideas to be recalled would include:

1. Many events are uncertain.
2. Two events may or may not be equally likely.
3. We can never be certain of the exact outcome of chance events.
4. There may be a pattern in large numbers of chance events. This pattern can help us estimate what is likely to happen if the events are repeated.
5. We can use experimental activities to check our estimates about chance events.

Show the spinner with dial $\frac{1}{2}$ red and $\frac{1}{2}$ blue. Ask questions such as:

> What part of the dial is red? $(\frac{1}{2})$ Blue? $(\frac{1}{2})$ If $\frac{1}{2}$ of the dial is red and $\frac{1}{2}$ is blue, is there one chance in two of getting red? (Yes.) Of getting blue? (Yes.)
>
> How could we change the dial so there would be a better chance of getting red? (Make more of the dial red.)
>
> What will happen to the chance of getting blue if we increase the chance of getting red? (Chance of getting blue will be less.)

Show the spinner with dial $\frac{1}{4}$ red and $\frac{3}{4}$ blue. Ask questions as before to show that there is 1 chance in 4 of getting red and 3 chances in 4 of getting blue. Also:

> Which of these spinners would you choose if you wanted to get blue on one spin? (The one with the largest blue area.)
>
> Which of these spinners would you choose if you hoped to get about 50 reds in 100 spins? (The first one--the one with equal amounts of red and blue.)

Show the spinner with dial $\frac{1}{3}$ red, $\frac{1}{3}$ blue, and $\frac{1}{3}$ yellow, and discuss similarities and differences among the three spinners. Bring out that there is 1 chance in 3 of getting red, 1 chance in 3 of getting blue, and 1 chance in 3 of getting yellow.

Summarize by helping children see, for example, that if $\frac{1}{2}$ of the dial is red means 1 chance in 2, then 1 chance in 2 means $\frac{1}{2}$ of the dial is red. Carry them through the same idea with the spinner with dial $\frac{1}{4}$ red and $\frac{3}{4}$ blue and the spinner with dial $\frac{1}{3}$ red, $\frac{1}{3}$ blue, and $\frac{1}{3}$ yellow.

Lead them to generalize that 1 chance in 2 of getting red means that the chance of red is $\frac{1}{2}$ and that, in this case, $\frac{1}{2}$ of the results are likely to be red. Allow sufficient discussion to develop understanding of this concept. Then ask such questions as:

> If a spinner dial is $\frac{2}{3}$ blue and $\frac{1}{3}$ red, what is the chance of getting red? ($\frac{1}{3}$, or 1 chance in 3.)
>
> What is the chance of getting blue? ($\frac{2}{3}$, or 2 chances in 3.)
>
> If the chance of getting red on a spinner is $\frac{1}{5}$, what part of the dial is red? ($\frac{1}{5}$.)
>
> What part is not red? ($\frac{4}{5}$.)
>
> Then what is the chance of getting a color other than red on such a spinner? ($\frac{4}{5}$, or 4 chances in 5.)

Continue until children understand the relationship between the chance of getting a given color and the fraction of the dial that is covered by that color. Then ask:

> If $\frac{1}{5}$ of a dial is red, what is the chance of getting red? (1 out of 5, or $\frac{1}{5}$.)
>
> If $\frac{4}{5}$ of a dial is red, what is the chance of getting red? (4 out of 5, or $\frac{4}{5}$.)
>
> If $\frac{5}{5}$ of a dial is red, what is the chance of getting red? (5 out of 5, or $\frac{5}{5}$, or 1. Children should generalize that when a result is certain to happen, the chance of that result is 1 out of 1, or $\frac{1}{1}$, or 1. Thus, if the chance of red equals 1, we know that red must occur

> and that red is the only result possible in this case.)
>
> Let us look at these ideas another way.
>
> If $\frac{4}{5}$ of a dial is red, then the chance of red is 4 out of 5, or $\frac{4}{5}$.
>
> If $\frac{1}{5}$ of a dial is red, then the chance of red is 1 out of 5, or $\frac{1}{5}$.
>
> Now, think about this. If no part of a dial is red, then what is the chance of red? (There is no chance of red because there is no red on the dial.)
>
> What number could we use to mean there is no chance of getting red? (Zero. Children should generalize that when a result cannot occur, the chance of that result is zero, or 0. Thus, if the chance of red equals 0, we know that red cannot occur and that the result, red, is impossible in this case.)

At this point, help children summarize these ideas. The summary should include the following points in the pupils' own words.

1. We can use fractions to compare the chances of different results.
2. If some result is certain to happen, we say the chance of that result is equal to one.
3. If some result cannot happen, we say the chance of that result is equal to zero.

Have pupils open their texts to page 38 and complete it as a class activity. Discuss it as necessary and then do pages 39, 40, and 41 together and have the children discuss their answers.

Do not go on to Pupil page 42 yet but introduce the last part of this lesson in a way similar to this:

> From our activities and discussions, we have discovered several ideas about chance. We can use numbers to describe the chance that some event may or may not occur. For instance, if an event is certain to occur, the chance that it will occur is

equal to 1. If an event cannot occur, the chance that it will occur is equal to 0. And, if an event is uncertain, the chance that it will occur is equal to some fraction between 0 and 1.

Mathematicians use the word probability in much the same way that we have used the word chance. (Write probability on the chalkboard.) We know that we can use a number to describe the chance that an event will occur. This number is called the probability of the event. Where we have said, "The chance of red is equal to $\frac{1}{2}$", a mathematician would say, and you may say, "The probability of red is equal to $\frac{1}{2}$."

In mathematics we use a few letters, numerals, and signs to stand for a big idea that we would otherwise have to use many words to explain. (Show spinner with $\frac{1}{2}$ red, $\frac{1}{2}$ blue dial.) If we asked a mathematician to describe the chance of red on this spinner, he would say, "The probability of red is $\frac{1}{2}$", and he would write $P(R) = \frac{1}{2}$. (Write "$P(R) = \frac{1}{2}$" on the chalkboard. Ask what the "R" represents.) From now on we will use just the first letter instead of writing out the name of the color.

Here are two cubes, one red and one white. If I place them in my pocket and then pick one without looking, how could we describe the chance that it will be white? That it will be red? Write on the board, $P(W) = \frac{1}{2}$, and, also, $P(R) = \frac{1}{2}$.

Suppose I add a blue cube so that there are a red, a white, and a blue. What is the probability of picking a blue? (John, will you write the number sentence on the board? [$P(B) = \frac{1}{3}$] A red? [$P(R) = \frac{1}{3}$] A white? [$P(W) = \frac{1}{3}$]

I will add the yellow cube. Now there are a red, a white, a blue, and a yellow. How could we describe the chance of picking a yellow? May, please write it on the board. [$P(Y) = \frac{1}{4}$] A blue? [$P(B) = \frac{1}{4}$]

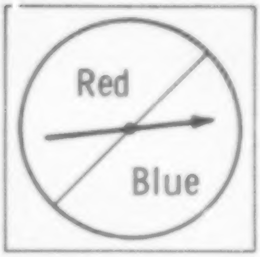
What is the probability of picking either a red or a white? [$P(R \text{ or } W) = \frac{2}{4} = \frac{1}{2}$] (Here we want either one of two of the four possible events.)

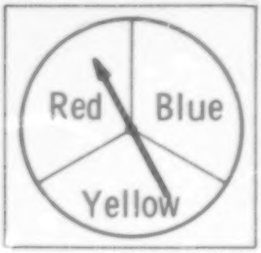
What is the probability of picking a green? [$P(G) = 0$. Whenever an event is impossible, its probability is 0.]

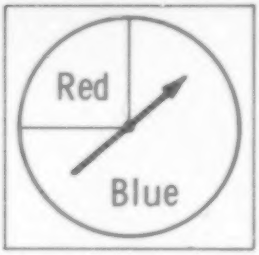
What is the probability of picking either a red, a white, a blue, or a yellow? [$P(R \text{ or } W \text{ or } B \text{ or } Y) = 1$. Whenever an event is certain to occur, its probability is 1.]

Now we have a way of using numbers to express probability. Let's complete Pupil pages 42 through 44. Many of you will be able to answer the Brain Teasers on page 45.

After the boys and girls have completed these pages, discuss their answers. On Pupil page 42, the answer expected to the last two items in the first column is 2 red cubes and 1 blue cube, but it could be 4 red cubes and 2 blue cubes, etc.







Look at these spinners.

You know that you can use fractions to compare the chances of different results.

Complete this table.

$\frac{1}{2}$ of dial red	means	1 chance in 2	means	Chance of red	= $\frac{1}{2}$
$\frac{1}{2}$ of dial blue	means	1 chance in 2	means	Chance of blue	= $\frac{1}{2}$
$\frac{1}{3}$ of dial red	means	1 chance in 3	means	Chance of red	= $\frac{1}{3}$
$\frac{1}{3}$ of dial blue	means	1 chance in 3	means	Chance of blue	= $\frac{1}{3}$
$\frac{1}{3}$ of dial yellow	means	1 chance in 3	means	Chance of yellow	= $\frac{1}{3}$
$\frac{1}{4}$ of dial red	means	1 chance in 4	means	Chance of red	= 1/4
$\frac{3}{4}$ of dial blue	means	3 chances in 4	means	Chance of blue	= 3/4
All of dial red	means	red is certain	means	Chance of red	= 1
none of dial red	means	red is impossible	means	Chance of red	= 0

Exercises - Lesson 6.

1. James spins the pointer of a spinner 100 times and gets 35 reds. Which of the following statements is most likely to be true?
   - (a) The dial of the spinner is all red.
   - (b) The dial of the spinner is one-half blue.
   - (c) The dial of the spinner is one-eighth red.
   - [✓ (d)] The dial of the spinner is one-third red.

2. Mary spins the pointer of a spinner 100 times and gets 25 red, 25 blue, and 50 yellow. Which of the following statements cannot be true?
   - (a) The dial of the spinner is one-fourth yellow.
   - (b) The dial of the spinner is one-third green.
   - (c) The dial of the spinner is one-fourth blue.
   - [✓ (d)] The dial of the spinner is all red.

3. A spinner has a dial that is one-third red, one-half white, and one-sixth blue. Which of the following cannot result from exactly 100 spins?
   - (a) 30 reds, 50 whites and 20 blues.
   - (b) 40 reds, 40 whites and 20 blues.
   - [✓ (c)] 50 reds, 5 whites and 10 blues.
   - (d) 60 reds, 40 whites and 0 blues.

4. You wish to get exactly 5 reds and 5 blues in 15 spins. Which of the following dials could not give this result?
   - [✓ (a)] One-half red and one-half blue.
   - (b) One-third red, one-third blue and one-third yellow.
   - (c) One-fourth red, one-fourth blue and one-half yellow.
   - (d) One-fifth red, two-fifths blue and two-fifths yellow.

5. In which of the following statements is the chance of red equal to $\frac{1}{4}$?
   - (a) One chance in two of red.
   - (b) Two chances in four of red.
   - (c) One chance in five of red.
   - [✓ (d)] Two chances in eight of red.

6. Which of the following spinners is likely to give about the same number of reds and yellows?
   - (a) One-half red, one-fourth yellow, one-fourth blue.
   - (b) One-third red, two-thirds yellow.
   - [✓ (c)] One-third red, one-third yellow, one-third blue.
   - (d) Four-fifths yellow, one-fifth red.

7. If the dial of a spinner is all red, we say the chance of red is equal to:
   - (a) any other chance.
   - (b) one chance in two.
   - (c) one-half.
   - [✓ (d)] one.

8. If the dial of a spinner is all blue, we say the chance of red is equal to:
   (a) one.
   [✓ (b)] zero.
   (c) one chance in one.
   (d) one-half.

9. The dial of a spinner is one-third red, one-third yellow, and one-third blue. Which of the following statements are true?
   [✓ (a)] Red, yellow, and blue are equally likely to occur.
   [✓ (b)] The chance of getting red is equal to $\frac{1}{3}$.
   [✓ (c)] One spin must result in either red or yellow or blue.
   [✓ (d)] The chance of getting green is equal to zero.

10. If the chance of red on a spinner is equal to zero, which of the following statements could be true?
   (a) The dial is all red.
   [✓ (b)] The dial is all blue.
   [✓ (c)] The dial has at least two colors.
   [✓ (d)] The dial has at least three colors.

11. Complete this table.

All of dial red	means	red is certain	means	Chance of red = 1	means	P(R) = 1
None of dial red	means	red is impossible	means	Chance of red = 0	means	P(R) = 0
$\frac{1}{2}$ of dial red $\frac{1}{2}$ of dial blue	means	1 chance in 2 of red 1 chance in 2 of blue	means	Chance of red = $\frac{1}{2}$ Chance of blue = $\frac{1}{2}$	means	P(R) = $\frac{1}{2}$ P(B) = $\frac{1}{2}$
$\frac{1}{4}$ of dial red $\frac{3}{4}$ of dial blue	means	1 chance in 4 of red 3 chances in 4 of blue	means	Chance of red = $\frac{1}{4}$ Chance of blue = 3/4	means	P(R) = $\frac{1}{4}$ P(B) = 3/4
$\frac{1}{3}$ of dial red $\frac{1}{3}$ of dial blue $\frac{1}{3}$ of dial yellow	means	1 chance in 3 of red 1 chance in 3 of blue 1 chance in 3 of yellow	means	Chance of red = 1/3 Chance of blue = 1/3 Chance of yellow = 1/3	means	P(R) = 1/3 P(B) = 1/3 P(Y) = 1/3
2 red cubes and 1 blue cube	means	2 chances in 3 of red 1 chance in 3 of blue	means	Chance of red = 2/3 Chance of blue = $\frac{1}{3}$	means	P(R = 2/3 P(B) = 1/3

12. A spinner has a dial which is evenly divided into red, white, and blue spaces. Write a number sentence that describes the chance of getting blue. P(B) = 1/3

13. Write a number sentence that answers the question, "What is the probability of yellow on the spinner in Problem 12 ?" P(Y) = 0.

14. In Problem 12, P(R) = 1/3.

15. Write this number sentence (about Problem 12) in words: $P(W) = \frac{1}{3}$.
The probability of getting white is 1/3.

16. A bag contains several marbles. Some are red, some white, and the rest blue. If you pick one marble without looking, the probability of red is $\frac{1}{3}$ and the probability of white is $\frac{1}{3}$. What is the probability of blue?
1/3

17. A bag contains one red marble, two white marbles, and three blue marbles. If you pick one marble without looking, what is the probability that the marble will be red? 1/6

18. In Problem 17, what is the probability that the marble will be white? 2/6 or 1/3

19. In Problem 17, what is the probability that the marble will be blue? 3/6 or 1/2.

20. In Problem 17, how many white marbles must be added to the bag to make the probability of white equal to $\frac{1}{2}$? 2

21. Write the following number sentence in symbols: "The probability of yellow is equal to three-fourths." P(Y) = 3/4.

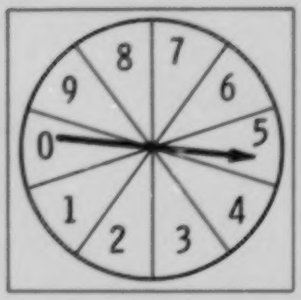
22. A wooden cube has a dot on two of its faces. If it is tossed on the floor, what is the probability that a face with a dot on it will be on the bottom when it stops rolling? 2/6 or 1/3

23. In Problem 22, what is the probability that a face without a dot will be on the bottom? 4/6 or 2/3

24. The dial of a spinner is divided into three colors: red, white, and blue. If $P(R) = \frac{1}{2}$ and $P(W) = \frac{1}{4}$, what is the probability of blue? 1/4

25. In Problem 24, is the probability of red greater than, less than, or equal to the probability of blue? greater than

The dial of this spinner is divided into 10 equal regions.

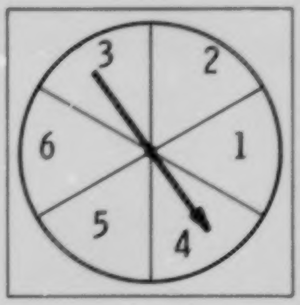


26. P(3) = 1/10 .

27. P(10) = 1/10 .

28. Is P(4) = P(8) ? yes

The dial of this spinner is divided into 6 equal regions.



29. P(2) = 1/6 .

30. P(5) = 1/6 .

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Brain Teasers

1. John has ten pairs of socks in a drawer. Nine pairs are red and one is blue. If he picks the socks one at a time without looking, how many socks must he pick to be sure he has two socks of the same color? 3

2. A bag contains several marbles. Some are red, some white, and the rest blue. If the probability of picking red is $\frac{1}{6}$ and the probability of picking white is $\frac{1}{3}$, what is the probability of picking blue? $\frac{1}{2}$

3. In Brain Teaser 2, what is the smallest number of marbles that could be in the bag? 6

4. In Brain Teaser 2, could the bag contain 48 marbles? Yes

5. In Brain Teaser 2, if the bag contains 4 red marbles and 8 white marbles, how many blue marbles does it contain? 12

LESSON 7

Complementary Events

Introduction.

Pupils first make a simple list of all the possible outcomes of a series of chance events. From this, probabilities are determined and then complementary events are introduced in an informal way through the use of number sentences.

The suggested procedure follows a logical sequence and offers many opportunities for practice in the development of number sentences as well as the addition and subtraction of common fractions. These opportunities can be shortened or extended as you wish.

Vocabulary: Complementary.

Materials: (1) Colored blocks or marbles (2 red, 2 blue).

(2) Two non-transparent bags or boxes.

Suggested Procedure:

Introduce the lesson by having pupils recall the following points from previous lessons:

1. Rational numbers and number sentences are used to express probabilities and statements about them.
2. An event that is certain to occur has a probability of 1. This can be written P(the event) = 1.
3. An event which cannot occur has a probability of 0. This can be written, P(the event) = 0.
4. No event can have a probability greater than 1 or less than 0.
5. Any event which may or may not occur, depending on chance, has a probability which can be expressed as a fraction between 0 and 1.

Allow sufficient discussion to clarify these five ideas. Use questions and illustrative materials as necessary.

Then show that you have two bags. Put one red block and one blue block in each bag. Call one the first bag and the other the second bag. Ask the class to think about an experiment in which we remove two blocks, one from the first bag and then one from the second bag. Ask:

> How many blocks are in the first bag? (2.) And what are their colors? (One red, one blue.)
>
> How many blocks are in the second bag? (2.) And what are their colors? (One red, one blue.)
>
> If we remove one block from the first bag, what color can it be? (Either red or blue.)
>
> Now if we remove one block from the second bag, what color can it be? (Either red or blue.)
>
> Let's list the possible outcomes. (List on the board):

The Possible Outcomes

1. Red Red
2. Red Blue
3. Blue Red
4. Blue Blue

Have children look at the list to see that each outcome includes two blocks: one from the first bag and one from the second bag and that there are 4 outcomes in all. (Red-Red; Red-Blue; Blue-Red; Blue-Blue.)

> How many of these four possible outcomes include two reds? (1.) What is the probability that both blocks will be red? ($\frac{1}{4}$.)
>
> How many of the outcomes include a red first and a blue second? (1.) What is the probability that the first block will be red and the second blue? ($\frac{1}{4}$.)
>
> What is the probability that the first block will be blue and the second red? ($\frac{1}{4}$.)
>
> What is the probability that both blocks will be blue? ($\frac{1}{4}$.)

> Think carefully about this. What is the probability that one block will be red and the other blue? ($\frac{2}{4}$ or $\frac{1}{2}$, since two of the four possible outcomes consist of one red and one blue block.)
>
> What is the probability that one of the blocks will be green? (Zero, since neither bag contains a green block.)

(List these probabilities on the chalkboard.)

$P(RR) = \frac{1}{4}$ $\qquad$ $P(G) = 0$

$P(RB) = \frac{1}{4}$ $\qquad$ $P(RB \text{ or } BR) = \frac{1}{2}$

$P(BR) = \frac{1}{4}$ $\qquad$ $P(RR \text{ or } RB \text{ or } BR \text{ or } BB) = 1$

$P(BB) = \frac{1}{4}$

Call attention to the fact that you have added a new probability, the probability of getting either two reds, one red and one blue, or two blues. Since this event includes all the possible outcomes, it is certain to happen. What is the probability in this case? (1.)

> Think about these events in another way. We know that the probability of getting two red blocks is $\frac{1}{4}$. What is the probability of not getting two red blocks? ($\frac{3}{4}$, since three of the four possible outcomes do not include two red blocks.)
>
> What is the probability of not getting a red first and a blue second? ($\frac{3}{4}$.)
>
> What is the probability of not getting a blue first and a red second? ($\frac{3}{4}$.)
>
> What is the probability of not getting two blues? ($\frac{3}{4}$.)
>
> What is the probability of not getting one of each color? ($\frac{1}{2}$.)
>
> What is the probability that we won't get two reds or two blues or one of each? (0, because we have to get something.)

> What is the probability of not getting a green block? (1. Since there are no green blocks we can be certain we will not get a green one.)

List on the board the probability that an event will not occur along with the probability that the same event will occur.

$P(RR) = \frac{1}{4}$ $\quad$ $P(\text{Not } RR) = \frac{3}{4}$

$P(RB) = \frac{1}{4}$ $\quad$ $P(\text{Not } RB) = \frac{3}{4}$

$P(BR) = \frac{1}{4}$ $\quad$ $P(\text{Not } BR) = \frac{3}{4}$

$P(BB) = \frac{1}{4}$ $\quad$ $P(\text{Not } BB) = \frac{3}{4}$

$P(G) = 0$ $\quad$ $P(\text{Not } G) = 1$

$P(RB \text{ or } BR) = \frac{1}{2}$ $\quad$ $P(\text{Not } RB \text{ and Not } BR) = \frac{1}{2}$

Point out that in the first column we have the probability that an event will occur and in the second column the probability that it will not occur. Ask what would happen if we find the sum of the two probabilities for each event. (The sum is 1 in each case.)

> Since each event must either occur or not occur, is it reasonable that the probability that it will occur plus the probability that it will not occur should be equal to one? (Yes, because one or the other is certain to happen and when something is certain to happen its probability is 1.)

(List on chalkboard.)

$$P(RR) + P(\text{Not } RR) = 1$$
$$P(RB) + P(\text{Not } RB) = 1$$
$$P(BR) + P(\text{Not } BR) = 1$$
$$P(BB) + P(\text{Not } BB) = 1$$
$$P(G) + P(\text{Not } G) = 1$$
$$P(RB \text{ or } BR) + P(\text{Not } RB \text{ and Not } BR) = 1$$

> The statements are true because we are talking about complementary events. Complementary events are opposites of each other; one of them must happen, but both cannot happen at the same time. To be complementary, two events must include all the possible outcomes. Therefore, the sum of their probabilities is always 1.

Explain that every event has a complement. For example, eating lunch today and not eating lunch today are complementary events. One of them must happen but both cannot happen today. Ask the children to think of others and allow time for discussion and clarification of this concept.

> Let us now look at these statements in another way. If P(RR) + P(Not RR) = 1, then how would we complete these statements?

(Write on chalkboard.)

1 - P(RR) = ______

1 - P(Not RR) = ______

Lead pupils to see that 1 - P(RR) = P(Not RR) and 1 - P(Not RR) = P(RR). Use examples, such as $\frac{1}{2} + \frac{1}{2} = 1$; therefore, $1 - \frac{1}{2} = \frac{1}{2}$, etc.

> Help me list the mathematical sentences that use this idea.

List on the chalkboard and have pupils state the last term for each sentence before you write it.

1 - P(RR) = P(Not RR) and 1 - P(Not RR) = P(RR)
1 - P(RB) = P(Not RB) and 1 - P(Not RB) = P(RB)
1 - P(BR) = P(Not BR) and 1 - P(Not BR) = P(BR)
1 - P(BB) = P(Not BB) and 1 - P(Not BB) = P(BB)
1 - P(G) = P(Not G) and 1 - P(Not G) = P(G)

Test some of these statements by using the correct numbers for the probabilities and seeing if the statements are true. Have pupils give correct numbers as you rewrite statements. For instance, 1 - P(RR) = P(Not RR) becomes $1 - \frac{1}{4} = \frac{3}{4}$. Continue until the idea is clearly established.

Pupil pages 46 and 47: These two pages give pupils an opportunity to check their understanding of complementary events. Questions 8, 9, and 10 help children realize that here one outcome is not affected by another outcome--what John picks does not influence what Paul picks, and Paul's pick does not depend upon what John picks.

Pupil page 48: For further activities, you may want to use the spinner on this page, for example, and ask questions about the probability that an outcome will be odd or even, ≤ 4 or > 4, etc. Other spinners could be used for this same purpose.

Exercises - Lesson 7.

John and Paul each have one white and one green marble. John picks one of his marbles without looking and then Paul picks one of his. The four possible outcomes are listed in the table below. Complete the table on the right to show the outcomes in a shorter way.

	John's Marble	Paul's Marble
1.	White	White
2.	White	Green
3.	Green	White
4.	Green	Green

	John's Marble	Paul's Marble
1.	W	W
2.	W	G
3.	G	W
4.	G	G

1. What is the probability that John picks a white marble? 1/2
2. What is the probability that Paul picks a white marble? 1/2
3. What is the probability that both boys pick white marbles? 1/4
4. What is the probability that both boys pick green marbles? 1/4
5. What is the probability that the boys pick a marble of the same color? 1/2

6. a. P(WW) = 1/4; P(Not WW) = 3/4.

b. P(WG) = 1/4; P(Not WG) = 3/4.

c. P(GW) = 1/4; P(Not GW) = 3/4.

d. P(GG) = 1/4; P(Not GG) = 3/4.

e. P(WG or GW) = 1/2; P(Not WG and Not GW) = 1/2.

7. a. P(GG) + P(WW) = $\frac{1}{4}$ + $\frac{1}{4}$ = 1/2.

b. P(GW) + P(WG) = 1/4 + 1/4 = 1/2.

c. P(WW) + P(Not WW) = 1/4 + 3/4 = 1.

8. a. If John picks a white marble, what is the probability that Paul will not pick a white marble? 1/2

b. Does John's outcome have any effect on Paul's outcome? no

9. a. If John picks a green marble, what is the probability that Paul also picks a green marble? 1/2

b. Does Paul's pick depend on what John picks? no

10. What is the probability that Paul will not pick a marble of the same color as John's? 1/2

A bag contains three marbles, one red, one white and one blue. Imagine that you choose one marble without looking.

11. P(R) = 1/3.

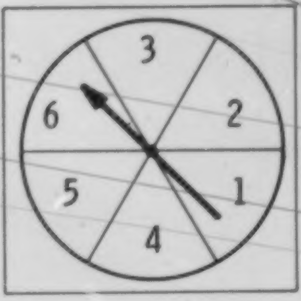
12. P(R or W) = 2/3.

13. P(Not B) = 2/3.

14. 1 - P(R) = 2/3.

15. P(R) + P(W) + P(B) = 1.

The dial of this spinner is divided into six equal regions.



16. P(3) = 1/6.

17. P(Not 3) = 5/6.

18. P(5 or 6) = 1/3.

19. P(1 or 3 or 4 or 5 or 6) = 5/6.

20. P(2) = 1/6.

21. Are the events in Problems 19 and 20 complementary events? Yes

22. 1 - P(3) = 5/6.

23. 1 - P(Not 3) = 1/6.

24. P(3) + P(Not 3) = 1.

LESSON 8

Counting Outcomes

Introduction.

In this lesson, pupils find probabilities by using simple "trees" and tables to count outcomes. Some pupils will learn that the probability of an outcome can be determined by finding the product of the probabilities of simple events. For example, consider the example in the previous lesson of two bags, each of which contains a red block and a blue block. If we pick a block from each bag, the four possible outcomes, with the block from the first bag listed first, are RR, RB, BR, BB, and they are all equally likely. So, by counting, the probability of getting a red from the first bag and a red from the second bag is $\frac{1}{4}$. The probability of getting a red from the first bag is $\frac{1}{2}$ and the probability of getting a red from the second bag is $\frac{1}{2}$. The probability of getting a red from both bags is $\frac{1}{4}$ which is the product of the probabilities for the separate bags.

For a single bag, there are two equally likely events. For two bags, there are 2×2, or 4, equally likely events. To find the total number of outcomes for two bags, we multiply the number of events for the first bag by the number of events for the second bag.

To find the probability of one of these outcomes, we multiply the probability for the first bag by the probability for the second bag. For instance, the probability of red from the first bag is $\frac{1}{2}$ and the probability of blue from the second bag is $\frac{1}{2}$. Therefore, the probability of the outcome red-blue is equal to $\frac{1}{2} \times \frac{1}{2}$, or $\frac{1}{4}$.

Vocabulary: Tree, table, row, column.

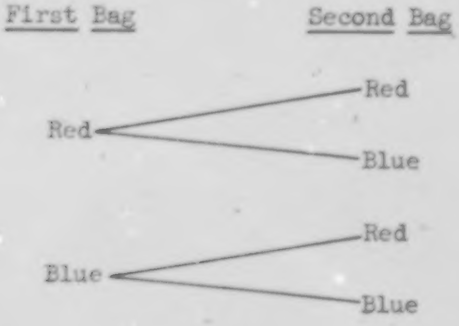
Materials: (1) Colored blocks or marbles (3 red, 3 blue)

(2) 3 non-transparent bags or boxes.

Suggested Procedure:

Show the two bags from the previous lesson and place one red block and one blue block in each bag. Call one the first bag and the other the second bag and ask questions for a rapid review: What is the probability of picking a red block from the first bag? ($\frac{1}{2}$.) From the second bag? ($\frac{1}{2}$.) What is the probability of picking a blue one from the first bag? ($\frac{1}{2}$.) From the second bag? ($\frac{1}{2}$.) What is the probability that both blocks will be red? ($\frac{1}{4}$.)

Suggest making a "tree" to show the possible outcomes. For instance, there are two possible events from the first bag and for each of these there are two possible events from the second bag. So there are 2 × 2, or 4, possible outcomes in all. Help pupils grasp this idea as you draw the tree on the board. Write the Red and Blue under First Bag and then draw the lines to the right, as shown, and write Red, Blue, Red, Blue under Second Bag:



Have pupils read the possible outcomes from the tree, going from left to right and top to bottom. (Red-Red, Red-Blue, Blue-Red, Blue-Blue.) You may want to use only the letters R and B on the tree.

> Of the 4 possible outcomes, how many include a red block from each bag? (1.) Then what is the probability of getting a red block from each bag? ($\frac{1}{4}$.)

State that sometimes a table is used to help find the possible outcomes. Show on the chalkboard:

		Second Bag	
		Red	Blue
First Bag	Red		
	Blue		

If the pupils have had no experience with tables, explain that the left side shows the color of the blocks taken from the first bag and that color will be indicated first in its row.

Second Bag

		Red	Blue
First Bag	Red	R	R
	Blue	B	B

The top of the table shows the color from the second bag, so that color is written second in the column below its name.

Second Bag

		Red	Blue
First Bag	Red	RR	RB
	Blue	BR	BB

Show that the possible outcomes are all given as in the trees, RR, RB, BR, BB. The table, like the tree, is read from left to right and top to bottom.

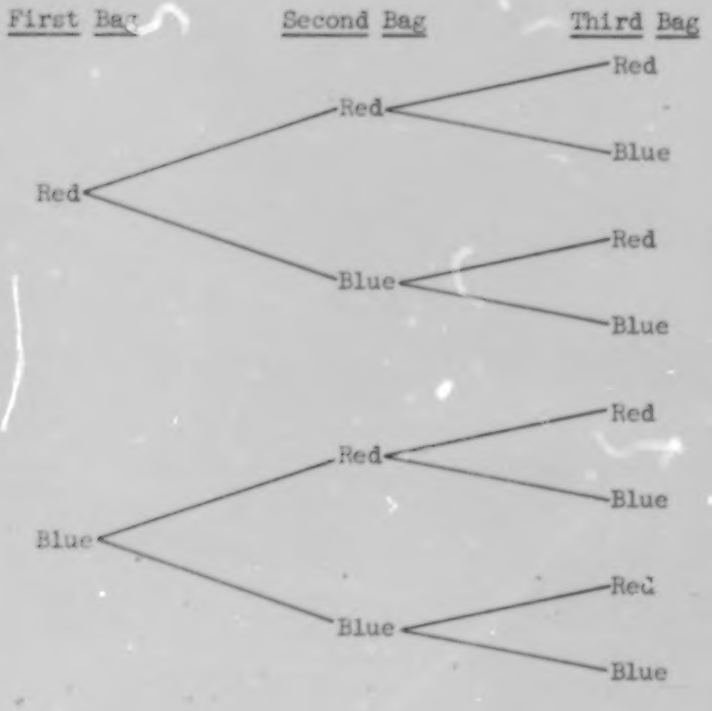
For some classes you may want to continue:

> Think about this. The probability of getting red from the first bag is $\frac{1}{2}$ and the probability of getting red from the second bag is $\frac{1}{2}$. The probability of getting red from both bags is $\frac{1}{4}$. There are two possible events from each of the two bags so we find the total number from both bags combined by multiplying: 2×2. Could this same idea help us to find the probability of each outcome when we pick a marble from both bags? (Yes, it seems to. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.)

Pupils who have had multiplication of common fractions will see this. Teacher judgment will determine how this will be treated and the degree of emphasis to be given to it.

In simple situations, probabilities can be determined by making a list of all possible outcomes and then counting to find the fractions that express them. However, when the number of outcomes is large, as it frequently is, listing and counting become impractical.

Introduce a third bag containing one red and one blue block. Have pupils state the outcomes for the three bags as you list them on the chalkboard. This will be an extension of the tree for two bags. To each of the outcomes for two bags you now add a red and a blue block from the third bag. There are then $2 \times 2 \times 2$, or 8, outcomes for 3 bags, compared with 2×2, or 4, outcomes for 2 bags.


First Bag
Second Bag
Third Bag
Red
Red
Red
Blue
Blue
Red
Blue
Blue
Red
Red
Blue
Blue
Red
Blue

On the board show pupils how to make a table for this. Remind them that they already have a table for two bags so it can be used in making a new table for 3 bags.

		Second Bag	
		Red	Blue
First Bag	Red	RR	RB
	Blue	BR	BB

		Third Bag	
		Red	Blue
First and Second Bags	RR	RRR	RRB
	RB	RBR	RBB
	BR	BRR	BRB
	BB	BBR	BBB

First, put the outcomes from the two bags in the first column of the new table. List them in order as you read from left to right and top to bottom. Stress that following a regular order helps us to show the possible outcomes systematically. Have pupils help you fill in the rest of the table, using letters for the names of the colors. Help the boys and girls read the tree and the table from left to right and top to bottom to see that the 8 possible outcomes are RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB.

Guide pupils to discover these points:

1. There are now 8 equally likely outcomes so the probability of any one is $\frac{1}{8}$. For example, P(Red-Red-Red) = $\frac{1}{8}$.

2. The probability of red for each separate bag is $\frac{1}{2}$. The probability of 3 reds, one from each bag, is $\frac{1}{8}$. For some classes this can be stated in a number sentence, as:

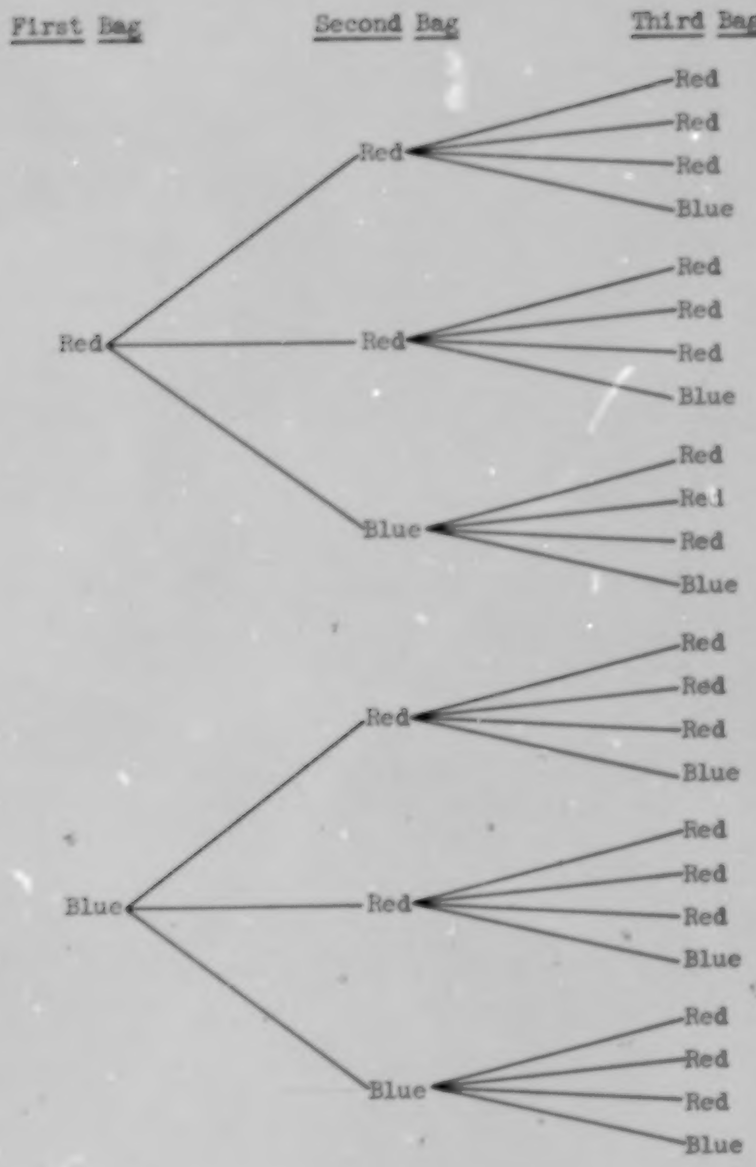
   P(Red from first bag) × P(Red from second bag) × P(Red from third bag) = P(Red-Red-Red) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

3. Each time we add a bag, we double the number of outcomes. For instance, with one bag there were two outcomes (R, B); with two bags there were four outcomes (RR, RB, BR, BB); and with three bags there are eight outcomes (RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB). The number of outcomes increases by powers of 2. The number of outcomes for one bag is 2; for two bags, 2^2; for three bags, 2^3, etc. For pupils who do not understand exponents, this can be shown as 2 for one bag; 2×2 for two bags; $2 \times 2 \times 2$ for three bags.

Use examples and discussion to establish these points. Pupils should understand that here we are interested in outcomes which include more than one simple event. To get three red blocks, we must get one from each bag so we are talking about three separate simple events, one for each bag, and we are interested in the probability that all three will occur. Each bag doubles the number of outcomes. When we increase the number of equally likely outcomes, we decrease the probability that any one outcome will occur. Some pupils will find this point rather obvious but all pupils should understand it, if possible.

4. To find the probability of 2 reds and 1 blue is harder; it is best done by counting possibilities. The blue block can be drawn from the first bag (so that we draw blue-red-red), or from the second bag (red-blue-red), or from the third (red-red-blue). We see from the tree or the table that each of these sequences appears just once among the possible 8 sequences. Therefore, a combination of 2 reds and 1 blue can happen in 3 ways of the total of 8, and so its probability is $\frac{3}{8}$.

If the maturity and insight of pupils permit, extend these ideas by discussing a situation in which the first bag contains 1 red and 1 blue block; the second bag, 2 red blocks and 1 blue block; the third bag, 3 red blocks and 1 blue block. In the second bag the two colors are not equally likely; the probability of red is $\frac{2}{3}$ and the probability of blue is $\frac{1}{3}$. Also, in the third bag, the two colors are not equally likely; the probability of red is $\frac{3}{4}$ and the probability of blue is $\frac{1}{4}$. If we draw a block from each bag, there are $2 \times 3 \times 4$ or 24 *possible* outcomes, all equally likely. The probability of drawing 3 reds, and the probability of drawing 3 blues, can be computed by multiplying or by counting. By multiplying, we get: probability of 3 reds is the probability of red from the first bag times the probability of red from the second bag times the probability of red from the third bag $= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$, or $\frac{6}{24}$, or $\frac{1}{4}$; the probability of 3 blues $= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$, or $\frac{1}{24}$. To compute these probabilities by counting, make use of a tree or a table, as is shown below. You may wish to have pupils state the various combinations as you construct the tree and complete a table on the board. This should help them see that constructing a tree or building a table can be tedious when there are more than a few outcomes. The mathematical computation of the probability is much faster. However, the tree and table have the advantage of being visual. Furthermore, to compute the probability of 2 reds and 1 blue, or of 1 red and 2 blues, we will have to use the tree or table; the mathematical method is too involved to be gone into at this stage.


First Bag
Second Bag
Third Bag
Red
Red
Red
Red
Blue
Red
Red
Red
Red
Blue
Blue
Red
Red
Red
Blue
Blue
Red
Red
Red
Red
Blue
Red
Red
Red
Red
Blue
Blue
Red
Red
Red
Blue

Second Bag

First Bag

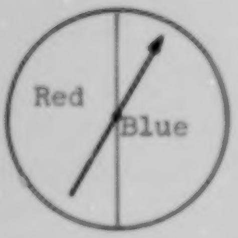
	Red	Red	Blue
Red	RR	RR	RB
Blue	BR	BR	BB

Third Bag

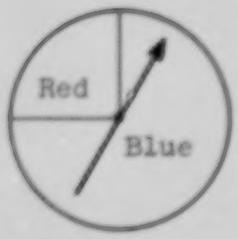
First and Second Bags

	Red	Red	Red	Blue
RR	RRR	RRR	RRR	RRB
RR	RRR	RRR	RRR	RRB
RB	RBR	RBR	RBR	RBB
BR	BRR	BRR	BRR	BRB
BR	BRR	BRR	BRR	BRB
BB	BBR	BBR	BBR	BBB

You can use the two spinners pictured below to illustrate these points in another way.



A



B

Spinner A has two outcomes and Spinner B has two outcomes. The total number of outcomes, if both are spun once, is 2×2, or 4. However, consider the following probabilities:

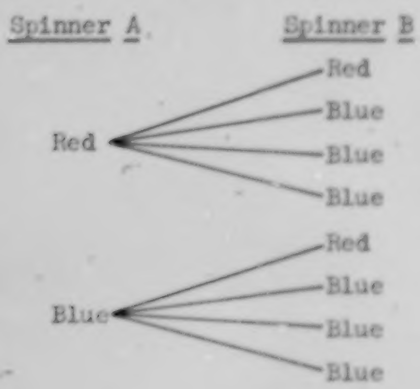
$$P(RR) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$P(BB) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$

$$P(RB) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$

$$P(BR) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

This list of the probabilities of all possible outcomes suggests that there must be 8, rather than 4, outcomes. This results from the fact that, while red and blue are equally likely on Spinner A, they are not equally likely on Spinner B. In fact, blue is 3 times as likely as red on Spinner B. As long as we stick to the mathematical probabilities for Spinner A and Spinner B, we can speak of two outcomes for Spinner B. However, if we construct a tree or table, we must think of Spinner B as being divided into 4 equal parts, 3 blue and 1 red, because there are 3 chances of blue to 1 chance of red on that spinner. This is illustrated in the tree and table below.



Spinner B

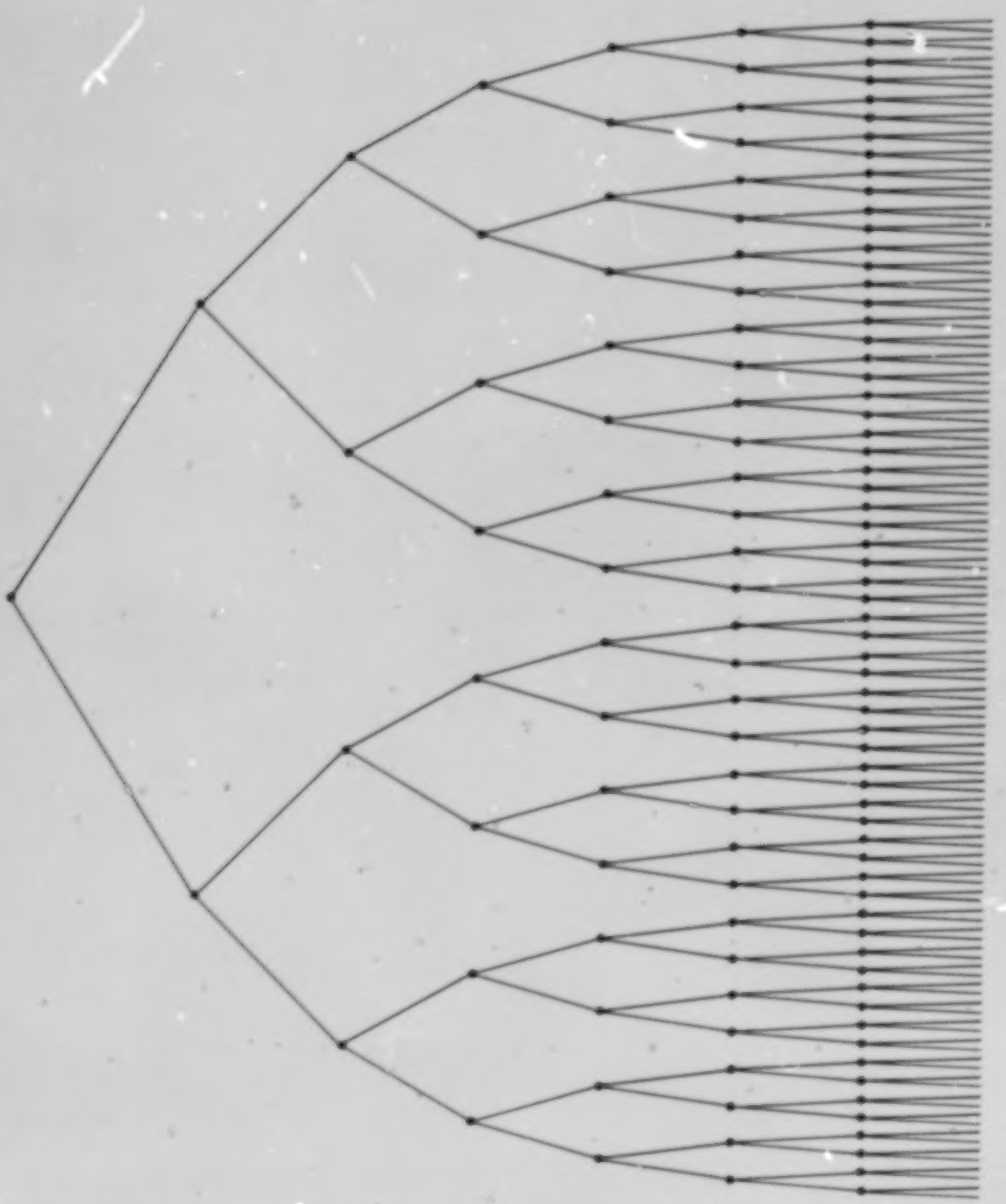
		R	B	B	B
Spinner A	R	RR	RB	RB	RB
	B	BR	BB	BB	BB

Pupil pages 49-52: Pupils should note the systematic manner in which the trees and tables are to be built, always working from left to right and top to bottom. The probabilities are easily determined by counting outcomes.

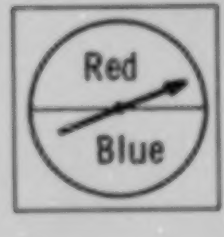
Pupil pages 53-55: As a further activity, pupils could make a table which shows the possible products of the numbers on two dice, or perhaps even three dice, and write questions concerning the various probabilities. Spinners with numbered dials could also be used for this type of activity. Some pupils might enjoy continuing Exercise 7 to include 4 or more coins.

Pupil pages 56 and 57: The symbols < (less than), > (greater than), and ≠ (not equal) may be unfamiliar to some pupils. Spinners with numbered dials could be used in many ways. For example, a table could be made of quotients by using two spinners or by spinning one spinner twice; or players could agree that they will divide the number obtained from a spin by a given number such as 6 and scores could be kept and tables made. The opportunities for all kinds of drill are almost limitless.

This form for a tree diagram might be useful for further activities for some pupils, both in this lesson and in the lesson on the Pascal Triangle.

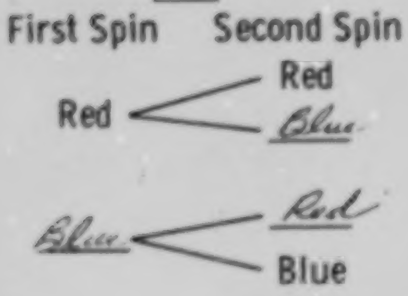


Exercises - Lesson 8.



1. Complete the tree diagram and the table to show the possible outcomes of two spins with this spinner.

Tree


First Spin
Second Spin
Red
Red
Blue
Blue
Red
Blue

Second Spin

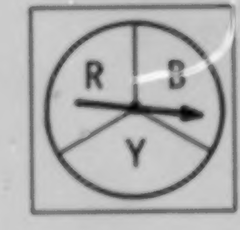
		Red	Blue
First Spin	Red	RR	RB
	BLUE	BR	BB

a. P(RR) = 1/4.

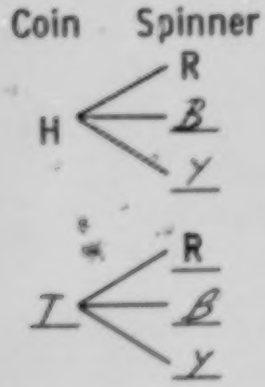
b. P(Not RR) = 3/4.

c. P(BR) = 1/4.

d. P(BB or RR) = 1/2.



2. Complete this tree diagram and the table. Show all the possible outcomes of the toss of a coin and one spin on this spinner. The dial is divided into three equal regions.


Coin
Spinner
H
R
B
Y
T
R
B
Y

Spinner

		R	B	Y
Coin	H	HR	HB	HY
	T	TR	TB	TY

a. P(HY) = 1/6.

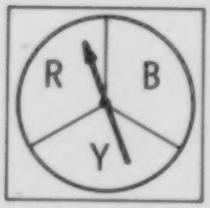
b. P(TB) = 1/6.

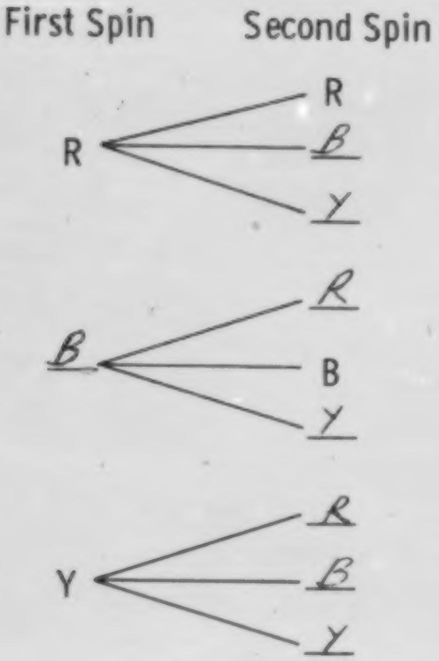
c. P(TR) = 1/6.

d. P(HB) = 1/6.

e. P(Not HB) = 5/6.

3. Complete this tree diagram and the table to show all the possible outcomes of two spins. The dial is divided into three equal regions.




First Spin
Second Spin
R
R
B
Y
B
R
B
Y
Y
R
B
Y

Second Spin

First Spin	R	B	Y
R	RR	RB	RY
B	BR	BB	BY
Y	YR	YB	YY

The total number of outcomes is 9.

a. P(RB) = 1/9.

b. P(YR) = 1/9.

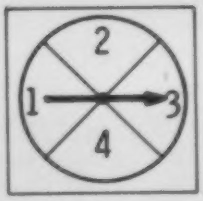
c. P(Not YR) = 8/9.

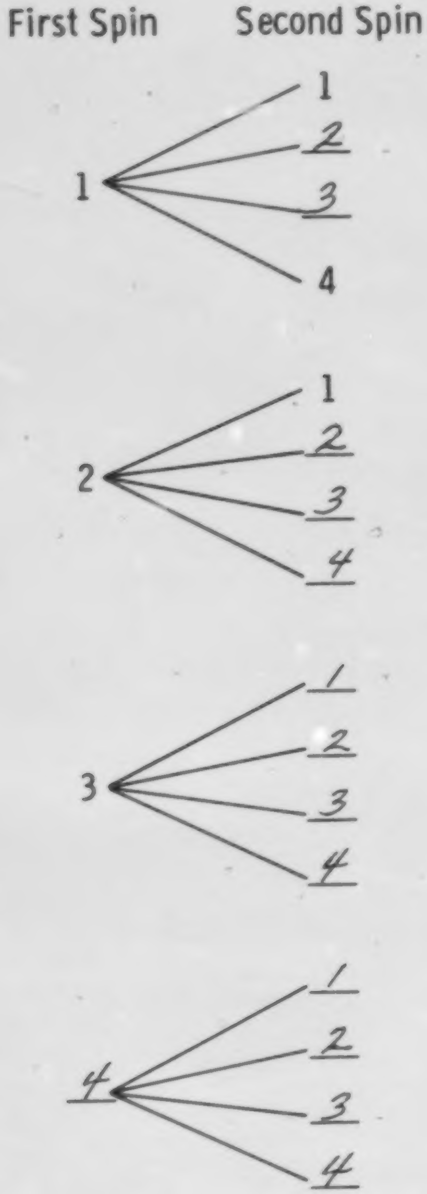
d. P(BY or RR) = 2/9.

e. P(RY or BY or YY) = 1/3.

f. How many possible outcomes are there if this spinner is spun three times? 27

4. Complete this tree diagram and the table to show all the possible outcomes of two spins with this spinner. The dial is divided into four equal regions.





Second Spin

First Spin	1	2	3	4
1	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

The total number of outcomes is 16.

a. P(3,4) = 1/16.

b. P(2,3 or 3,2) = 1/8.

c. P(Not 4,2) = 15/16.

d. P(Not 1,2 and Not 3,2) = 7/8.

e. P(Two odd numbers) = 1/4.

f. P(Not two odd numbers) = 3/4.

g. P(Three odd numbers) = 0.

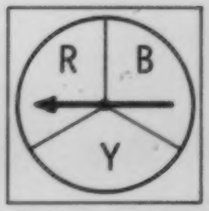
h. P(Two odd numbers or two even numbers) = 1/2.

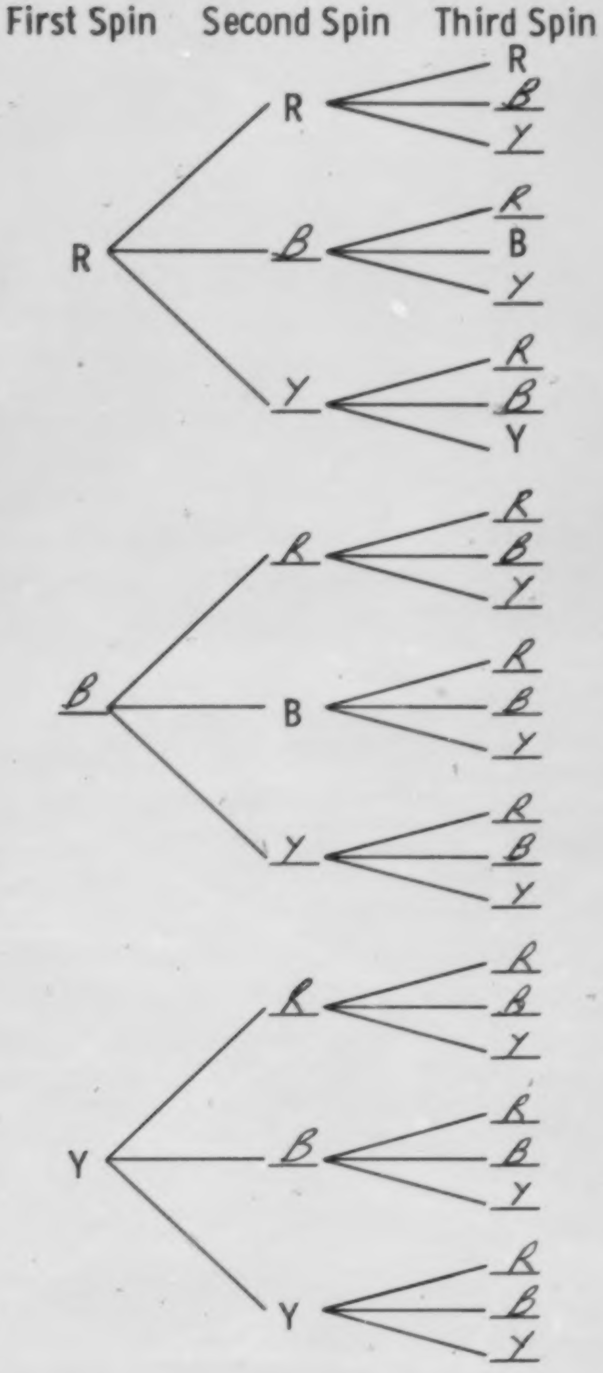
i. P(At least one 1) = 7/16.

j. P(Not more than one odd number) = 3/4.

k. This spinner is spun four times. How many possible outcomes are there? 256

5. Complete this tree diagram and the tables to show all the possible outcomes of three spins. The dial is divided into three equal regions.




First Spin
Second Spin
Third Spin
R
R
R
B
Y
B
R
B
Y
Y
R
B
Y
B
R
R
B
Y
B
R
B
Y
Y
R
B
Y
Y
R
R
B
Y
B
R
B
Y
Y
R
B
Y

Second Spin

First Spin	R	B	Y
R	RR	RB	RY
B	BR	BB	BY
Y	YR	YB	YY

Third Spin

First and Second Spins	R	B	Y
RR	RRR	RRB	RRY
RB	RBR	RBB	RBY
RY	RYR	RYB	RYY
BR	BRR	BRB	BRY
BB	BBR	BBB	BBY
BY	BYR	BYB	BYY
YR	YRR	YRB	YRY
YB	YBR	YBB	YBY
YY	YYR	YYB	YYY

a. P(RBB) = 1/27.

b. P(YRB) = 1/27.

c. P(Not BRB) = 26/27.

d. P(YYR) = 1/27.

e. P(YRR or RRR) = 2/27.

f. The probability that the colors will be the same on three spins is 1/9.

6. Complete the table to show all the possible sums of the dots on two dice.

Number on Second Die

Number on First Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a. One way to get a sum of 7 is to get a 1 on the first die and a 6 on the second die. We can write this as (1,6). There are five more ways to get a sum of 7. List them below.

(2,5), (3,4), (4,3), (5,2), (6,1).

b. How many entries are there in the table? 36

c. How many possible entries are there when you toss two dice?

36

d. Of the entries in the table, how many are 6's ? 5

e. What is the probability of getting a sum of 6 when two dice are tossed? 5/36

f. How many of the entries are odd numbers? 18

g. What is the probability of getting a sum that is an odd number? 1/2

h. How many of the sums are either 5's or 9's? 8

i. What is the probability that the sum will be either 5 or 9? 2/9

j. (1) P(sum = 3) = 1/18.

(2) P(sum = 8) = 5/36.

(3) P(sum = 12) = 1/36.

(4) P(sum = 2) = 1/36.

(5) P(sum = 11) = 1/18.

(6) P(sum = 2 or sum = 12) = 1/18.

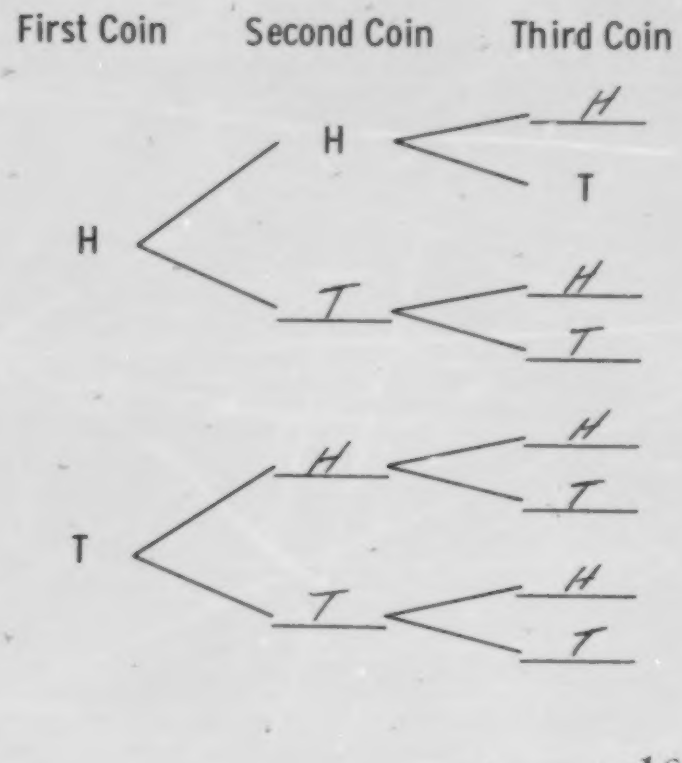
(7) P(sum = 6 or sum = 8) = 5/18.

(8) P(sum = 5 or sum = 9) = 2/9.

(9) P(sum ≠ 7) = 5/6.

(10) P(sum > 9) = 1/6.

7. Fill in the tree diagram and the tables to show all the outcomes when three coins are tossed. Use it to answer questions a through j.



		Second Coin	
		H	T
First Coin	H	HH	HT
	T	TH	TT

		Third Coin	
		H	T
First and Second Coins	HH	HHH	HHT
	HT	HTH	HTT
	TH	THH	THT
	TT	TTH	TTT

55

a. What is the total number of outcomes when three coins are tossed? 8

b. How many of these outcomes include three heads? 1

c. What is the probability of getting three heads when three coins are tossed? 1/8

d. How many of these outcomes include two heads and one tail? 3

e. What is the probability of getting two heads and one tail when three coins are tossed? 3/8

f. What is the probability of getting three tails when three coins are tossed? 1/8

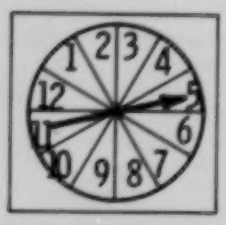
g. What is the probability of getting at least one tail when three coins are tossed? 7/8

h. What is the probability that you will not get three heads or three tails when you toss three coins? 3/4

i. What is the probability that you will get heads on the first coin? 1/2

j. (1) P(3 heads) = 1/8.
(2) P(2 heads, 1 tail) = 3/8.
(3) P(Not 3 heads) = 7/8.
(4) P(3 heads or 3 tails) = 1/4.
(5) P(At least 2 tails) = 1/2.
(6) P(No heads) = 1/8.
(7) P(3 tails) = 1/8.
(8) P(No tails) = 1/8.
(9) P(At least 1 head) = 7/8.
(10) P(At least 1 head or 1 tail) = 1.

8. Complete the table to show all the possible sums of numbers on two spins. The dial is divided into twelve equal regions.



Second Spin

First Spin	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	4	5	6	7	8	9	10	11	12	13	14
3	4	5	6	7	8	9	10	11	12	13	14	15
4	5	6	7	8	9	10	11	12	13	14	15	16
5	6	7	8	9	10	11	12	13	14	15	16	17
6	7	8	9	10	11	12	13	14	15	16	17	18
7	8	9	10	11	12	13	14	15	16	17	18	19
8	9	10	11	12	13	14	15	16	17	18	19	20
9	10	11	12	13	14	15	16	17	18	19	20	21
10	11	12	13	14	15	16	17	18	19	20	21	22
11	12	13	14	15	16	17	18	19	20	21	22	23
12	13	14	15	16	17	18	19	20	21	22	23	24

a. How many entries are represented in this table? 144

b. Of these entries, how many are 9's ? 8

c. What is the probability of getting a sum of 9 on two spins? 1/18

d. P(sum = 5) = 1/36. e. P(sum = 12) = 11/144.

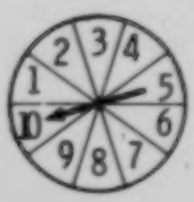
f. P(sum is odd) = 1/2. g. P(sum > 18) = 7/48.

h. P(sum < 4) = 1/48. i. P(sum ≠ 2) = 143/144.

j. Which sum has the greatest probability? 13

k. Which two sums have the least probability? 2 and 24

9. Complete the table to show the possible products of numbers on two spins. The dial is divided into ten equal regions.



Second Spin

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

a. How many entries are represented in this table? 100

b. How many times is there a product equal to 16 ? 3

c. What is the probability that a product equals 16 ? 3/100

d. P(24) = 1/25 .

e. P(27) = 1/50 .

f. P(even number) = 3/4 .

g. P(16 < product < 30) = 19/100 .

h. P(multiple of 5) = 9/25 .

i. List the products which have a probability of $\frac{1}{100}$:

1 , 25 , 49 , 64 , 81 , 100

LESSON 9

Both, And; Either, Or

Introduction.

In this lesson pupils use trees and tables to list the possible outcomes of making two draws from a bag which contains three different-colored marbles.

The first marble drawn is returned to the bag before the second draw is made. By counting outcomes, probabilities are then determined, with attention directed, for example, to both red on the first draw and green on the second. Then by counting outcomes, the probability is found for either red on the first draw or green on the second draw or both red on the first draw and green on the second draw. Thus, the both, and--either, or situations are introduced.

Next, pupils make a tree and a table to list the possible outcomes of two draws from a bag which contains 3 marbles of different colors but now the first marble is not returned to the bag after it is drawn. Then, by counting outcomes, several examples are given in which both, and probabilities and either, or probabilities are determined.

Vocabulary: Both, and; either, or.

Materials: (1) Colored blocks or marbles (1 red, 1 green, 1 white).

(2) Non-transparent bag.

Suggested Procedure:

Show the bag and place in it a red, a green, and a white marble. Ask what the probability is of drawing a red marble. ($\frac{1}{3}$.)

> If the marble that is drawn on the first draw is returned to the bag, what is the probability of green on the second draw? ($\frac{1}{3}$.) Can we find the probability of getting both a red marble on the first draw and a green marble on the second draw?

Ask the pupils to turn to page 58 in their texts and to make the table and the tree diagram to show all the possible outcomes of two draws from this bag. Remind them that the marble drawn on the first draw is returned to the bag before the second draw is made. Answer the questions on this page as a class activity, drawing attention to the both, and probability and making sure that the pupils comprehend what both, and means.

Work through Pupil page 59 as a class activity, using the table and tree from the previous page. Help the boys and girls understand the either, or situation and discuss the exercises at the bottom of the page. Some pupils might discover during the discussion of the probability of either red on the first draw or green on the second draw or both red on the first and green on the second, that:

P(Either R on first or G on second or both) = P(R) + P(G) - P(Both)

If you think your class should see this, take time to develop it, but otherwise, do not point it out to the pupils.

Show the bag and again put the red, green, and white marbles in it.

> Let's make two draws from this bag but this time we will not replace the marble we get on the first draw. Now, what is the probability of both red on the first draw and green on the second draw?

Have pupils complete the table and the tree on page 60 in their text to find the possible outcomes. They must be careful because now the first marble drawn is not replaced so the table and tree will not have entries in every space. Work together to complete this page and to answer the questions on page 61, helping children to use their table or tree to find the answers. Compare the probability of the "both, and" statement in Exercise 4 with the "either, or" statement in Exercise 8.

Pupil pages 62 and 63: These pages can be completed and discussed as a class activity or can be done independently by the pupils.

Pupil page 64: This game gives boys and girls practice in division and drill on decimal equivalents of fractions in addition to being a good "Probability Game". It would seem that player B has the advantage in this game but a table of outcomes which gives the first digit of the quotient of $\frac{A}{B}$ shows this isn't true:

Spinner A

Spinner B	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	5	1	1	2	2	3	3	4	4
3	3	6	1	1	1	2	2	2	3
4	2	5	7	1	1	1	1	2	2
5	2	4	6	8	1	1	1	1	1
6	1	3	5	6	8	1	1	1	1
7	1	2	4	5	7	8	1	1	1
8	1	2	3	5	6	7	8	1	1
9	1	2	3	4	5	6	7	8	1

The number of 1st digits are:

1 - 29
2 - 13
3 - 8
4 - 6
5 - 7
6 - 6
7 - 5
8 - 6
9 - 1
Total 81

Player A wins on 1, 2, or 3 so he wins on 50 of the 81 possible outcomes. Player B wins on 4, 5, 6, 7, 8, or 9 so he wins on 31 of the 81 possible outcomes.

This game could be played by finding the quotient of any two numbers. Another further activity which might interest some pupils would be to note the first digits of addresses in a telephone book. They could open the book at random and jot down the first digit of 100 addresses. How many begin with a 1 ? a 2 ? a 3 ? Do over half the addresses begin with one of these three numbers? Do almost one-third begin with a 1 ? Probably the answers to these two questions would be "yes".

Lesson 9 - "Both, and ; either, or"

A bag contains 3 marbles, one red, one green, and one white.

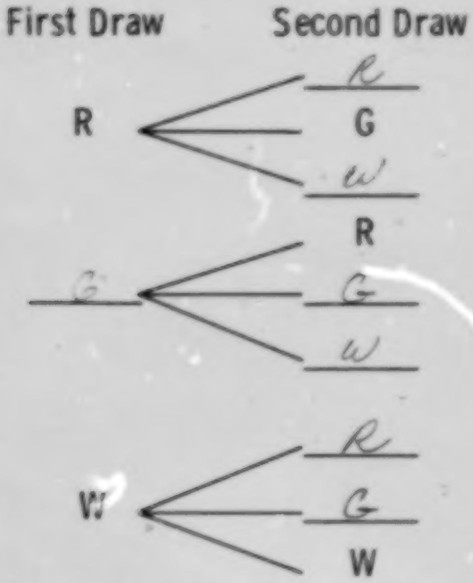
Let's draw a marble, note its color, and return it to the bag. Shake the bag and draw again.

$$P(\text{red on first draw}) = \frac{1}{3}.$$

$$P(\text{green on second draw}) = \frac{1}{3}.$$

We want to find the probability of getting both a red marble on the first draw and a green marble on the second draw. Complete the table and tree diagram of the possible outcomes.

		Second Draw		
		Red	Green	White
First Draw	Red	R R	RG	RW
	Green	GR	GG	GW
	White	WR	WG	WW



Both the table and the tree show 9 possible outcomes. Only one of them is Red on the first draw and Green on the second draw (RG).

P(RG) = P(Both red on first and green on second) = 1/9.

P(WR) = 1/9. P(WW) = 1/9.

Let's think of this same bag with 3 marbles, one red, one green, and one white.

But now, what is the probability of either red on the first draw or green on the second draw or both red on the first and green on the second?

Look at the table or the tree diagram we just made. There are 3 possible outcomes in which the first draw is red:

RR, RG, and RW.

There are also 3 outcomes in which the second draw is green:

RG, GG, and WG.

Notice that you listed one of the outcomes twice. Which one? RG

Therefore, there are only 5 outcomes with either red on the first draw or green on the second, or both red on the first and green on the second.

They are RR, RG, RW, GG, and WG.

We can count to see that

P(either red on first or green on second or both) = 5/9.

1. P(either R on first or W on second or both) = 5/9.
2. P(either G on first or G on second or both) = 5/9.
3. P(R on second or W on first or G on first) = 7/9.
4. P(R on first or G on first or W on first) = 9/9 or 1

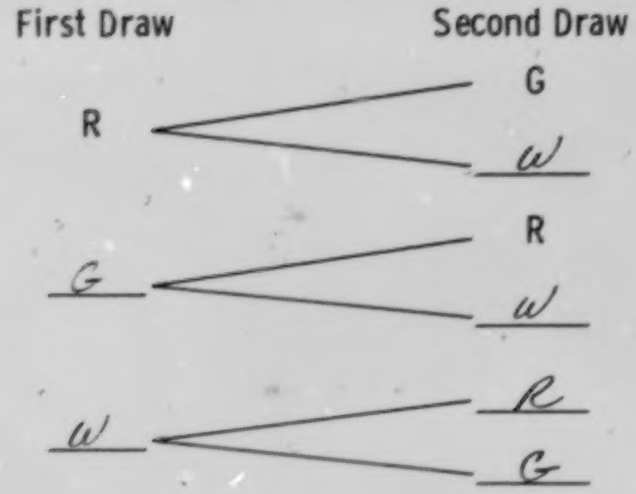
Think of the same bag and three marbles, red, green, and white.

This time we do not replace the first marble before we draw the second marble. What is the probability of red on the first draw and green on the second draw ? ________

Complete this table to show the outcomes. Remember that if we draw a red on the first draw and do not replace it, it cannot be drawn on the second draw. So R,R is impossible. Is G,G possible? no

		Second Draw		
		Red	Green	White
First Draw	Red	impossible	R G	R W
	Green	G R	impossible	G W
	White	W R	W G	impossible

A tree diagram would also show the possible outcomes. As you complete it, be careful to include only the possible outcomes.



1. How many possible outcomes are there? 6

2. What is the probability of red on the first draw? 1/3

3. What is the probability of green on the second draw? 1/3

4. What is the probability of both red on the first draw and green on the second draw? (Look for RG in the table.) 1/6

5. What is the probability of white on the first draw? 1/3

6. What is the probability of both white on the first draw and white on the second draw? 0

7. List the outcomes which show a red on the first draw, or a green on the second draw, or both. RG, RW, W[illegible]

8. P(either red on first or green on second or both) = 1/2.

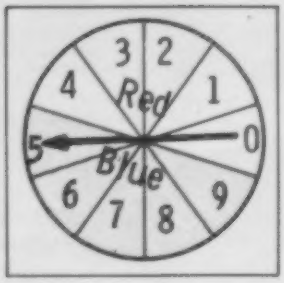
9. P(either green on first or white on second or both) = 1/2.

10. P(WW) = 0.

11. The probability that white will be drawn on either the first draw or second draw or both is 2/3. Does the phrase "or both" affect this statement? no Why or why not? WW is impossible

12. The probability that white will not be drawn at all on the two draws is 1/3.

Exercises - Lesson 9.



Use this spinner. The dial is divided into ten equal regions.

1. What is the probability of red? 1/2
2. What is the probability of blue? 1/2
3. What is the probability of a 7 ? 1/10
4. P(5) = 1/10.
5. P(even number) = 1/2.
6. P(odd number) = 1/2.

Let's try our phrases "both, and" and "either, or" to find:

7. the probability of both red and even;
8. the probability of either red or even or both.

Complete this table to help you find these probabilities.

Numbers

		0	1	2	3	4	5	6	7	8	9
Colors	Red	R,0	R,1	R,2	R,3	R,4					
	Blue						B,5	B,6	B,7	B,8	B,9

(Check your spinner; is a red 7 possible? No)

9. How many possible outcomes are there? 10
10. List the outcomes which are both red and even:

R,0, R,2, R,4.

11. Now answer Question 7. P(both red and even) = 3/10.

12. List the outcomes which are red:

R,0 , R,1 , R,2 , R,3 , R,4 .

13. How many outcomes are red? 5

14. List the outcomes which are even:

R,0 , R,2 , R,4 , B,6 , B,8 .

15. How many outcomes are even? 5

16. How many outcomes in Problems 12 and 14 are listed twice? 3

(R,0 , R,2 + R,4)

17. Now answer Question 8. P(either red or even or both) = 7/10 .

18. P(both red and 2) = 1/10 .

19. P(both blue and 4) = 0 .

20. P(either blue or even or both) = 8/10 .

21. P(either red or odd or both) = 8/10 .

22. P(both red and < 4) = 4/10 .

23. P(red or > 4) = 10/10 or 1

24. P(blue or > 4) = 1/2 .

25. P(blue or red) = 10/10 or 1

26. P(both blue and red) = 0 .

27. P(both red and > 4) = 0 .

Brain Teaser - Quacky Quotients

This is a game for two players. You need two spinners numbered 1 to 9. (You can use the spinners labeled 0-9 and spin again whenever either spinner stops on a zero, or you can cut out a new dial, divided into 9 equal parts, and place it under the pointer of any spinner.)

Each player spins once. The one with the larger number is called player A and the other is player B.

Each player spins his spinner. The number on player A's spinner is divided by the number on player B's spinner.

If the first digit of the quotient is a 1 or 2 or 3 (or .1 or .2 or .3), player A wins a point. If the first digit of the quotient is 4 or 5 or 6 or 7 or 8 or 9 (or .4 or .5 or .6 or .7 or .8 or .9), B wins a point. Thus A wins on 3 of the 9 possibilities while B wins on 6 of the 9 possibilities.

Is it a fair game? Play it and then decide. Here are examples:

A spins an 8, B spins 3. $\frac{8}{3} = 2. + \ldots$ So A wins.

A spins a 7, B spins a 9. $\frac{7}{9} = .7 + \ldots$ B wins.

A spins a 1, B spins a 4. $\frac{1}{4} = .25$. A wins.

A spins a 3, B spins 5. $\frac{3}{5} = .6$. B wins.

Spin 20 times and see who wins. Play several games before you decide if this is a fair game. Then make a table to show all the possible quotients in this game. From this table you can see which player has an advantage. What is the probability that player A will win a point on one set of spins? 50/81 that player B will win on one set of spins? 31/81 Why could we not use 0 on the dial in this game? ______________________________

____

LESSON 10

The Pascal Triangle

Introduction.

This lesson introduces the Pascal triangle in an intuitive way. Pupils are not expected to develop a formula for it but are encouraged to discover as many properties of the triangle as they can. In later grades, they will have the opportunity for more rigorous mathematical study. You might want to review the material in the Mathematical Comments before teaching this lesson.

Tables and trees are used to picture the possibilities for outcomes of heads or tails when coins are tossed. These possibilities are listed and from them the Pascal triangle is introduced. In the section, Things to Do at Home, several activities are suggested which will enable boys and girls to compare their data from experiments with that which they might expect from the pattern of the Pascal triangle. If time permits, this section would be very worthwhile to complete in class.

This lesson has many values.

1. The Pascal triangle contains a number of patterns which are a part of many different probability problems. For example, as we add one more element, we increase the number of outcomes by one more power of 2; for example, 2 outcomes for 1 coin (2^1); 4 outcomes for 2 coins (2^2); 8 outcomes for 3 coins (2^3); 16 outcomes for 4 coins (2^4); etc.
2. Pupils can, perhaps for the first time, appreciate a beautiful example of symmetry in mathematics; for example, the left half of each line duplicates the right half of the same line.
3. Many opportunities are afforded to discover patterns and relationships.
4. Pupils learn to read a different type of display and to see a new arrangement from which they can extract information.

5. More and more insight can be gained from the Pascal triangle as pupils become more sophisticated in mathematics.

6. The Pascal triangle is an excellent example of the value of classifying and organizing information in a meaningful and useful way.

Perhaps the last time you yourself were concerned with this concept was in a high school algebra class. You will enjoy helping children learn intuitively what you may have learned more formally.

Vocabulary: Pascal triangle.

Materials: (1) a straightedge for each pupil

(2) a straightedge for use on the chalkboard

Suggested Procedure:

The lesson might be introduced:

> We have found the possible outcomes for various combinations of simple events. We used trees and tables to list the outcomes. Today, we'll use trees and tables to find some interesting patterns that were noted by a mathematician, Pascal, about 300 years ago.

Ask pupils to turn to page 65 in their texts and complete the work. Discuss the activity of tossing a coin twice:

> In this activity, how many possible outcomes are there in all? (4.)
>
> How many outcomes are there in which both coins are heads? (1.)
>
> What is the probability of 2 heads? ($\frac{1}{4}$.)
>
> How many outcomes are there in which one coin is a head and the other is a tail? (2.)
>
> What is the probability of 1 head and 1 tail? ($\frac{2}{4}$, or $\frac{1}{2}$.)

How many outcomes are there in which both coins are tails? (1.)

What is the probability of 2 tails? ($\frac{1}{4}$.)

Call attention to the numbers 1 2 1, and write them on the board.

Discuss in a similar way the tree and table for 3 tosses of a coin, calling attention to the numbers 1 3 3 1, and write them on the board:

$$\begin{matrix} & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{matrix}$$

Pupils can then complete page 66. Discuss it as you did with two tosses and three tosses and write 1 4 6 4 1 on the board:

$$\begin{matrix} & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{matrix}$$

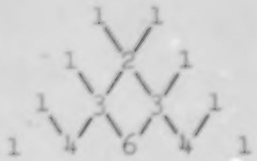
Ask how many different outcomes there are when 1 coin is tossed, (2), and what these are. (1 head, 1 tail.) Suggest that you will put this on the board and also add a little more information:

	Outcomes
1 coin	$^{1}{}_{1H,0T}$ $^{1}{}_{0H,1T}$
2 coins	$^{1}{}_{2H,0T}$ $^{2}{}_{1H,1T}$ $^{1}{}_{0H,2T}$
3 coins	$^{1}{}_{3H,0T}$ $^{3}{}_{2H,1T}$ $^{3}{}_{1H,2T}$ $^{1}{}_{0H,3T}$
4 coins	$^{1}{}_{4H,0T}$ $^{4}{}_{3H,1T}$ $^{6}{}_{2H,2T}$ $^{4}{}_{1H,3T}$ $^{1}{}_{0H,4T}$

Ask how many different outcomes there are when 1 coin is tossed. (2.) 2 coins. (4.) 3 coins. (8.) 4 coins. (16.) Ask if the sum of the numbers in each row, for example, 1 1 in row 1 and 1 2 1 in row 2, is equal to the number of outcomes. (Yes.)

Explain that this is a part of the Pascal triangle, a pattern noticed by the seventeenth century mathematician, Pascal. The table would look more like a triangle if the vertex at the top were given, but the idea of 1 possibility (no heads or tails) if no coins are used may be confusing to children. Pupils can now turn to page 67. Discuss this with them, helping them to discover, for example:

1. Each entry is the sum of the two numbers nearest to it on the line above. For example,



2. Each line is symmetrical; it can be read from right to left exactly as it can be read from left to right.

3. The sum of the numbers in any horizontal row is always a power of 2 .

4. The sum of the numbers in any one horizontal row is equal to the total number of outcomes; for example, row 3, for 3 coins, includes the numbers 1 3 3 1 . Their sum is 8 which is the total number of possible outcomes when 3 coins are tossed.

Ask the children to try to complete the fifth row and then to check their thinking on page 68. Work with them to complete this page. The triangle should be carefully completed with the number of heads and tails noted. Then ask how each new row would relate to an experiment with coins: How many coins would be needed; how many chances there would be of getting various combinations; how the number of outcomes relates to the sum of the numbers in the row, etc. Explain that although this particular sheet, page 67, shows the number of heads and tails we might expect when tossing a number of coins at one time, or one coin a number of times, it can be used for experiments involving two equally likely and mutually exclusive events (the occurrence of either excludes that of the other). It is interpreted, for example:

> If we toss 10 coins, we'd look at line 10 . It tells us that there are 2^{10} or 1,024 ways in which these 10 coins might fall. There is only 1 chance in 1024 that all 10 coins will be heads. $P(10\ H) = \frac{1}{1024}$. Also, there is only 1 chance in 1024 that all 10 coins will be tails. Other information from this line is:
>
> If we toss 10 coins,
>
> > there are 10 chances in 1024 that there will be 9 heads and 1 tails.
>
> $$P(9H, 1T) = \frac{10}{1024} .$$

there are 45 chances in 1024 that there
will be 8 heads and 2 tails.

$$P(8H,2T) = \frac{45}{1024} .$$

there are 120 chances in 1024 that there
will be 7 heads and 3 tails.

$$P(7H,3T) = \frac{120}{1024} .$$

As a class activity, complete the exercises on page 69. Pupils can discover by studying the triangle that when we know the number of heads, we also know the number of tails. For example, using 8 coins, the probability of 6 heads is $\frac{28}{256}$ and the probability of 2 tails also is $\frac{28}{256}$. Also, the probability of 6 tails is $\frac{28}{256}$ and the probability of 2 heads is $\frac{28}{256}$.

Some children might want to use line 10 to figure out what line 11 would be:

Line 10 2^{10} 1 10 45 120 210 252 210 120 45 10 1

\ / \ / \ / \ / \ / \ / \ / \ / \ / \ /

Line 11 2^{11} 1 11 55 165 330 462 462 330 165 55 11 1

The use of the Pascal triangle in making selections is introduced in the Pupil Exercises. For example, if there are 8 boys, how many different ways can you choose 5 of them to be on a committee? Since there are 8 boys, look at row 8. Then look for 5H,3T which represents a committee of 5, with 3 not on the committee. The number associated with (5H,3T) is 56. So there are 56 different ways in which a committee of 5 can be chosen from a group of 8 people. Note that when you choose a committee of 5, you automatically choose a group of 3, which does not serve on the committee. So you could have looked to the right of 8 for (3H,5T) and again 56 would be found as the number of different ways a committee of 3 can be formed from 8 people. Thus, there are 56 different ways a committee of 5 can be formed from 8 people, and 56 different ways a committee of 3 can be formed from 8 people.

Perhaps this seems complicated but it really isn't. It's just more difficult to write than it is to do! Let's try another example. From 5 books, how many different ways can you select 3 books? First, note the number of things, (5), so look at the fifth row. Next, look to the right for the number of books you wish to select, (3H,2T) or (2T,3H). The number associated with this is 10. So there are 10 ways in which you can select 3 books from 5 books. Also, there are 10 ways in which you can select 2 books from 5 books because when you select 3 books, you are at the same time rejecting 2 books--and this amounts to a selection, in a sense!

How many ways can you select 5 children to help you if you must choose from 7 children? Is your answer 21? It should be.

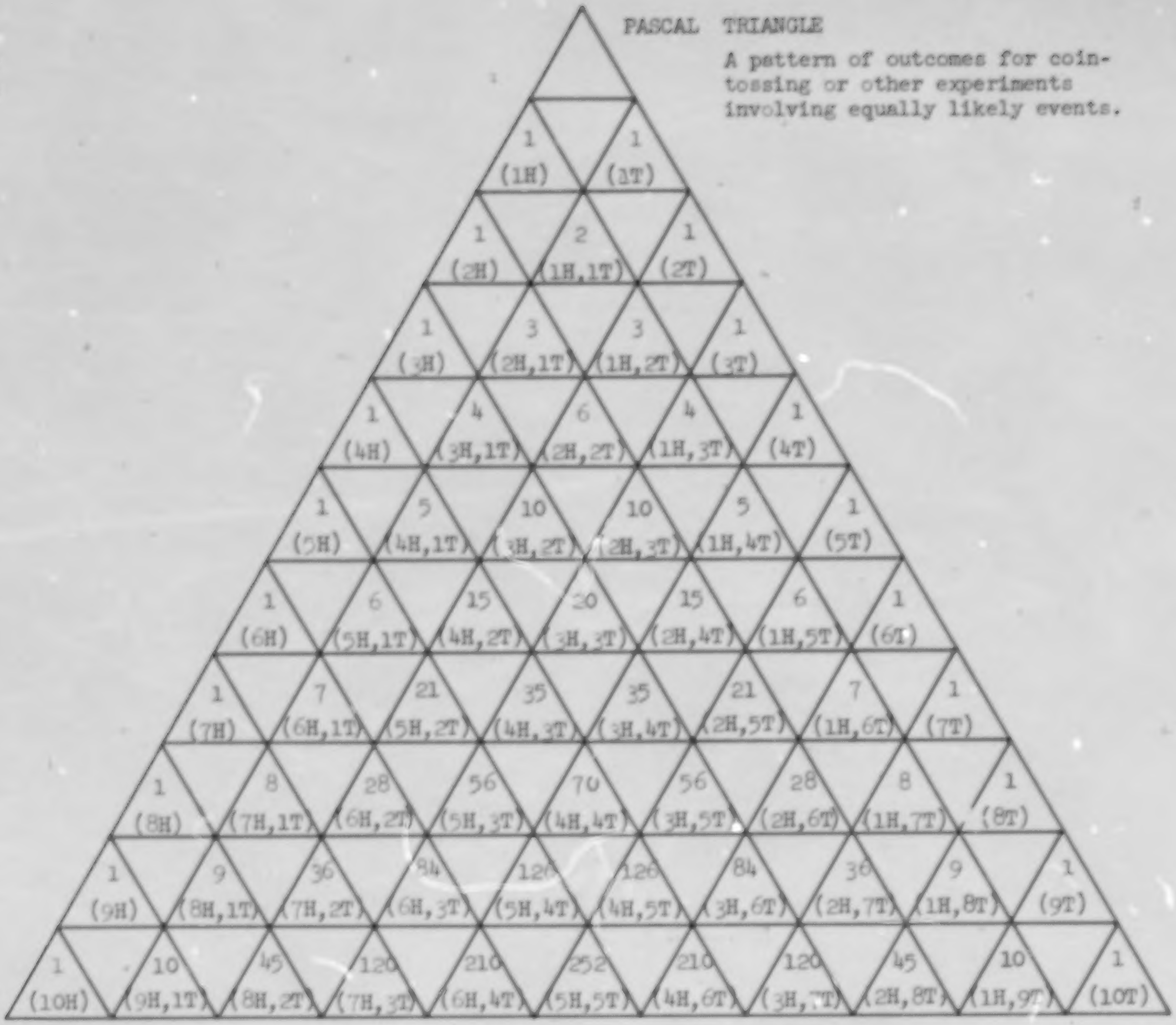
There are an apple, an orange, a banana, and a peach. How many different ways can you choose 3 of them? (Remember, there are 4 things. The answer is 4 ways.) How many ways can you select 2 of these? (6.)

A copy of the first 10 rows of the Pascal triangle is given on the next page.

Pupil pages 70 and 71: You may need to help pupils get started on page 70 by answering the questions at the top of the page as a class activity. Once pupils see how to use the triangle to make selections, they can answer these exercises quite rapidly.

Pupil pages 72-76: Discuss these pages in class when pupils report the data for the activities. This is an excellent opportunity to clarify ideas about expected results. Bar graphs could be made from the totals reported by several pupils and these could be quite useful in comparing actual results with expected results.

Number of Things	Total Number of Outcomes
1	2^1 (2)
2	2^2 (4)
3	2^3 (8)
4	2^4 (16)
5	2^5 (32)
6	2^6 (64)
7	2^7 (128)
8	2^8 (256)
9	2^9 (512)
10	2^{10} (1024)


1 (1H) 1 (1T)
1 (2H) 2 (1H,1T) 1 (2T)
1 (3H) 3 (2H,1T) 3 (1H,2T) 1 (3T)
1 (4H) 4 (3H,1T) 6 (2H,2T) 4 (1H,3T) 1 (4T)
1 (5H) 5 (4H,1T) 10 (3H,2T) 10 (2H,3T) 5 (1H,4T) 1 (5T)
1 (6H) 6 (5H,1T) 15 (4H,2T) 20 (3H,3T) 15 (2H,4T) 6 (1H,5T) 1 (6T)
1 (7H) 7 (6H,1T) 21 (5H,2T) 35 (4H,3T) 35 (3H,4T) 21 (2H,5T) 7 (1H,6T) 1 (7T)
1 (8H) 8 (7H,1T) 28 (6H,2T) 56 (5H,3T) 70 (4H,4T) 56 (3H,5T) 28 (2H,6T) 8 (1H,7T) 1 (8T)
1 (9H) 9 (8H,1T) 36 (7H,2T) 84 (6H,3T) 126 (5H,4T) 126 (4H,5T) 84 (3H,6T) 36 (2H,7T) 9 (1H,8T) 1 (9T)
1 (10H) 10 (9H,1T) 45 (8H,2T) 120 (7H,3T) 210 (6H,4T) 252 (5H,5T) 210 (4H,6T) 120 (3H,7T) 45 (2H,8T) 10 (1H,9T) 1 (10T)

PASCAL TRIANGLE

A pattern of outcomes for coin-tossing or other experiments involving equally likely events.

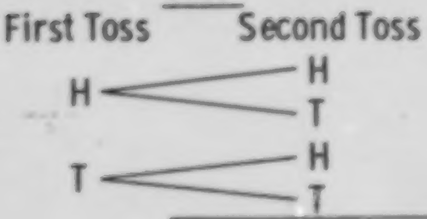
The Pascal Triangle - Lesson 10.

List some of the activities we have tried which have had two equally likely outcomes.

a. Spinner $\frac{1}{2}$ red, $\frac{1}{2}$ blue ;
b. coins
c. choosing 1 of 2 cards
d. noting if top face on a die is odd or even
e. choosing 1 of 2 cubes
f. odd or even numbers on a spinner with equa odd & even areas:

We will make tree diagrams for one of these activities, tossing a coin. Show the possible outcomes for two tosses.

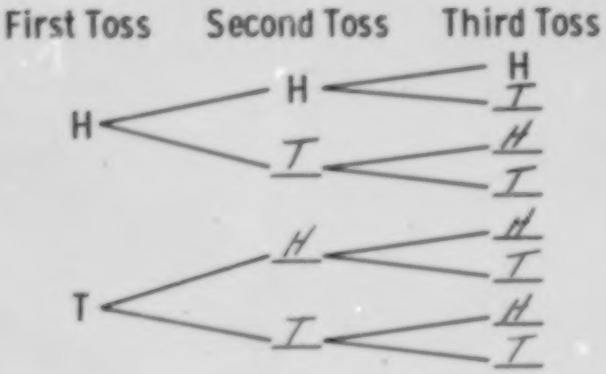
Tree


First Toss
Second Toss
H
H
T
T
H
T

Table

2H, 0T	1H, 1T	0H, 2T
H H	H T	T T
	T H	
Number of outcomes 1	2	1

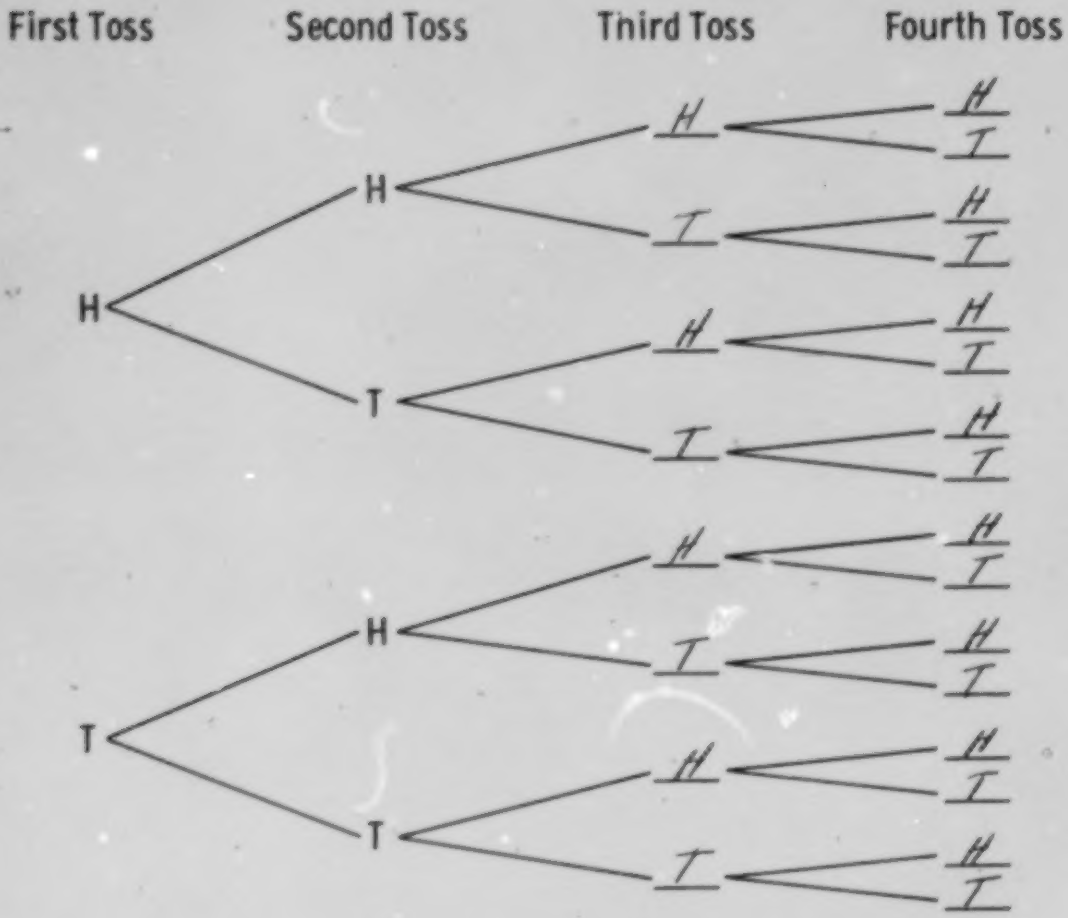
Complete the tree and table for 3 tosses.


First Toss
Second Toss
Third Toss
H
H
H
T
T
H
T
T
H
H
T
T
H
T

3H, 0T	2H, 1T	1H, 2T	0H, 3T
H H H	H H T	HTT	TTT
	HTH	THT	
	THH	TTH	

Number of outcomes	1	3	3	1

Now make one more tree (for four tosses).


First Toss
Second Toss
Third Toss
Fourth Toss
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T
H
T

List the outcomes in the table.

	4H, 0T	3H, 1T	2H, 2T	1H, 3T	0H, 4T
	HHHH	HHHT	HHTT	HTTT	TTTT
		HHTH	HTHT	THTT	
		HTHH	HTTH	TTHT	
		THHH	THHT	TTTH	
			THTH		
			TTHH		
Number of outcomes	1	4	6	4	1

Let's organize our data from the tables into a triangular display.

___First Toss	1 1H,0T	1 0H,1T									
___Second Toss	1 2H,0T	2 1H,1T	1 0H,2T								(Look at the first table.)
___Third Toss	1 3H,0T	3 2H,1T	3 1H,2T	1 0H,3T							(From the second table.)
___Fourth Toss	1 4H,0T	4 3H,1T	6 2H,2T	4 1H,3T	1 0H,4T						
___Fifth	1 5H,0T	5 4H,1T	10 3H,2T	10 2H,3T	5 1H,4T	1 0H,5T					
___Sixth	1 6H,0T	6 5H,1T	15 4H,2T	20 3H,3T	15 2H,4T	6 1H,5T	1 0H,6T				
___Seventh	1 7H,0T	7 6H,1T	21 5H,2T	35 4H,3T	35 3H,4T	21 2H,5T	7 1H,6T	1 0H,7T			
___Eighth	1 8H,0T	8 7H,1T	28 6H,2T	56 5H,3T	70 4H,4T	56 3H,5T	28 2H,6T	8 1H,7T	1 0H,8T		
___Ninth	1 9H,0T	9 8H,1T	36 7H,2T	84 6H,3T	126 5H,4T	126 4H,5T	84 3H,6T	36 2H,7T	9 1H,8T	1 0H,9T	
___Tenth	1 10H,0T	10 9H,1T	45 8H,2T	120 7H,3T	210 6H,4T	252 5H,5T	210 4H,6T	120 3H,7T	45 2H,8T	10 1H,9T	1 0H,10T

Look at our display. Do you see a pattern? yes Do you see several different patterns? yes

151 Can you complete the fifth row (5 tosses of a coin) without making a tree or a table? yes

Go back to the fifth row and try it. 152

Your fifth row should be

$$1_{5H,0T}\ 5_{4H,1T}\ 10_{3H,2T}\ 10_{2H,3T}\ 5_{1H,4T}\ 1_{0H,5T}$$

Did you write this row so that it continues the triangle? Yes

This pattern was noticed by a 17th-Century mathematician, Blaise Pascal. So it is named the Pascal Triangle.

Go back to the triangle and complete rows 6 through 10. Don't rush through it. Place a ruler so that it touches the ends of the 1's. Draw a light line down toward the "Tenth Toss". Do the same on both sides; it will help you keep your display in order. Always work neatly. Your results will be used to answer some questions.

Now go back to the first row of the triangle. Write in, just to the left of the words "First Toss", the number 2. The 2 stands for the total number of outcomes for one toss of a coin.

Just to the left of "Second Toss", write a 4 to show it represents 4 outcomes for 2 tosses of a coin. What should you write to the left of "Third Toss" ? 8 Complete by writing the total number of outcomes for each row. (The 7th row is 128 and the 10th row is 1024.)

Do you also see a pattern to this column of numbers? Yes

What is the pattern? Each number is twice as large as the numbers preceding it. The number of outcomes doubles on every toss. That is, if 1 coin is tossed there are 2 possible outcomes, if 2 coins are tossed there are 4 possible outcomes, if 3 coins are tossed there are 8 possible outcomes, etc.

Use the Pascal Triangle to answer these questions.

1. How many outcomes are there for 2 tosses? 4
2. How many outcomes of exactly 2 heads are there in 2 tosses? 1
3. In 2 tosses, P(2H) = 1/4.
4. In 3 tosses, P(3H) = 1/8.
5. In 7 tosses, P(7H) = 1/128.
6. In 7 tosses, P(7T) = 1/128.
7. In 4 tosses, how many possible outcomes are there? 16
8. In 4 tosses, how many outcomes can you expect to be 2H,2T ? 6
9. In 4 tosses, P(2H,2T) = 3/8.
10. In 4 tosses, P(3H,1T) = 1/4.
11. In 6 tosses, P(4H,2T) = 15/64.
12. In 6 tosses, P(2H,4T) = 15/64.
13. In 10 tosses, P(8H,2T) = 45/1024
14. In 10 tosses, P(2H,8T) = 45/1024.
15. In 10 tosses, P(5H,5T) = 252/1024
16. If you toss 10 coins at one time, is the probability of 4 heads and 6 tails more likely or less likely than 7 heads and 3 tails?

    more likely

Brain Teaser. If you toss 15 coins at one time, the greatest probability would be for 8 heads, 7 tails, and for 7 heads, 8 tails.

Let's see how we can use the Pascal Triangle to help us answer other problems. Look at the third row.

If the question were, "How many different combinations of exactly 2 Heads are there?", you would say 3.

This triangle can be used to answer many questions.

Use the third row again. There are 3 people. How many different combinations of 2 people can you choose from 3 people? (Look at 2H,1T.)

3

Exercises - Lesson 10.

Use your Pascal Triangle to answer these questions.

1. From 4 people, how many committees of 2 can be chosen? (Look at 4th row, 2H,2T). 6
2. From 4 people, how many committees of 3 can be chosen? 4
3. From a group of 7 people, there are:
   a. How many different committees of 3 people? 35
   b. How many different committees of 4? 35
   c. How many different committees of 1? 7
   d. How many different committees of 6? 7
   e. How many different committees of 7? 1

Exercises - Lesson 10 (continued).

4. a. In how many different ways can a committee of 6 be chosen from a group of 8 pupils? 28

   b. In how many different ways can a committee of 7 be chosen from a group of 8 pupils? 8

   c. In how many different ways can a committee of 8 be chosen from a group of 8 pupils? 1

5. The teacher asks for 3 volunteers to go with her to the library. However, 9 pupils volunteer.

   a. In how many different ways can the teacher select the 3 to go with her? 84

   b. If the teacher could select 4 from the 9 pupils, how many different groups could there be? 126

6. There are 5 flavors of ice cream: chocolate, vanilla, strawberry, black walnut, and peach.

   a. In how many different ways can you choose 2 of these flavors? 10

   b. In how many different ways can you choose 3 of these flavors? 10

   c. In how many different ways can you choose 4 of these flavors? 5

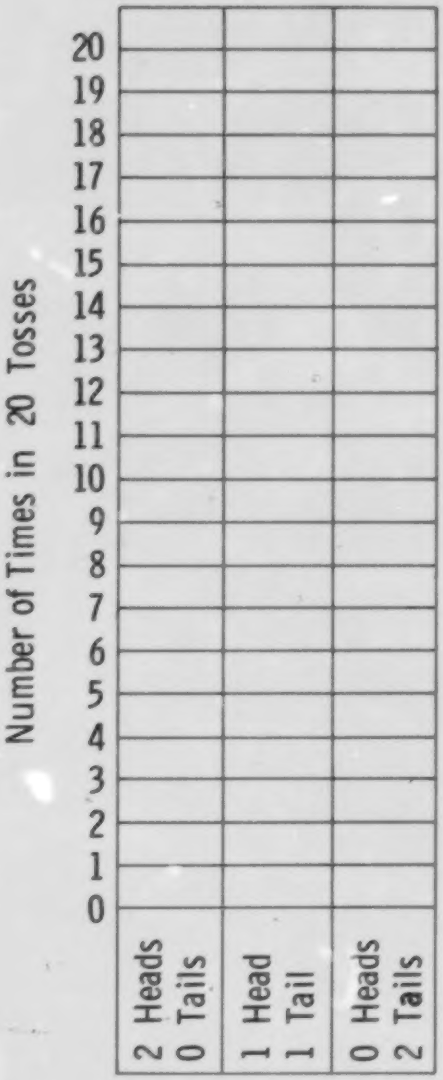
Things To Do At Home - Lesson 10.

1. The Pascal Triangle can be used to show the different possible outcomes in making selections. It is a pattern and we expect results similar to it. To compare the actual results with the Pascal Triangle, toss two coins and record the outcomes on the chart and on the graph.

CHART

Toss Number	2 Heads 0 Tails	1 Head 1 Tail	0 Heads 2 Tails
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
Totals			

BAR GRAPH



a. Did 2 heads occur about as often as 2 tails? ________

b. Did 1 head, 1 tail occur about twice as often as 2 heads? ________

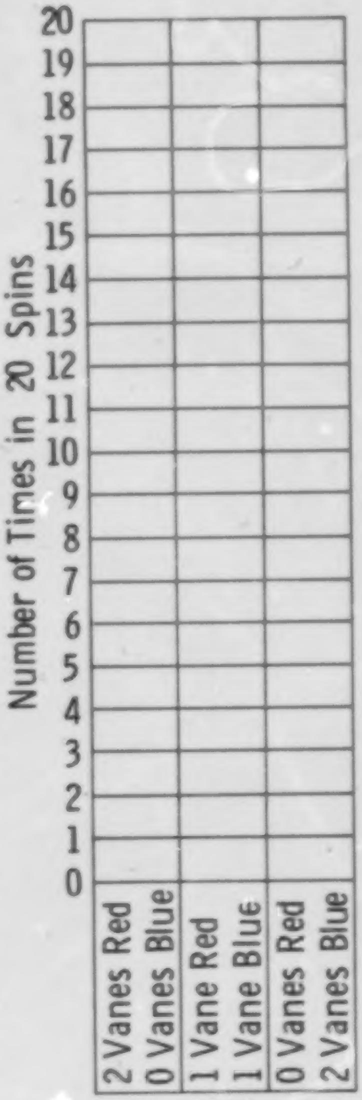
c. Were your results about what you thought they might be? ________

2. Make a windmill with two vanes. See the Appendix to learn how to make it. Color each vane red on one side and blue on the opposite side. Blow on the vanes and then place the windmill on a flat surface to see which vanes are up. Mark the chart to show how the vanes stop: both vanes red, one vane red and one vane blue, or both vanes blue. Show the totals on the graph.

CHART

Spin Number	2 Vanes Red 0 Vanes Blue	1 Vane Red 1 Vane Blue	0 Vanes Red 2 Vanes Blue
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
Totals			

BAR GRAPH



a. Would you expect the results of this experiment to be about the same as with the coin tossing experiment with two coins? Yes

b. Were your results about what you expected them to be? ________

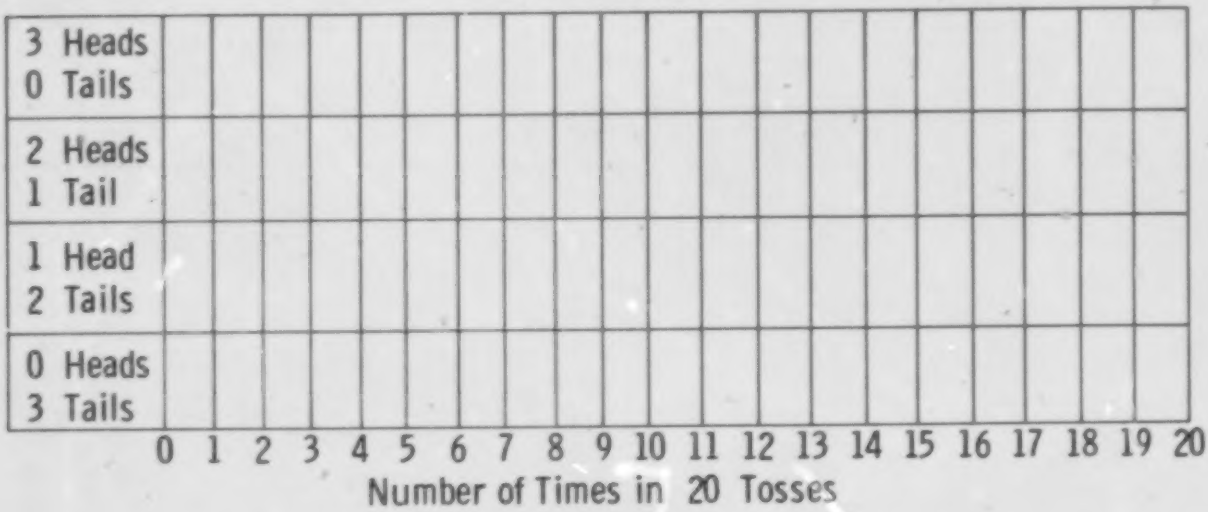
c. Circle your answer. About how many times would you expect an outcome of 1 vane red and 1 vane blue if you did this experiment 1000 times? 400 [✓ 500] 600

3. Toss three coins. Mark the chart and complete the graph from the totals. Compare your results with row 3 of the Pascal Triangle.

CHART

Toss Number	3 Heads 0 Tails	2 Heads 1 Tail	1 Head 2 Tails	0 Heads 3 Tails
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
Totals				

BAR GRAPH



a. From the Pascal Triangle, do you expect 2 heads, 1 tail, will occur about as often as 2 tails, 1 head? Yes

b. Were your results about what you expected? ________

c. If you tossed 3 coins 600 times, about how many times would you expect all 3 coins to land with the heads up? 75

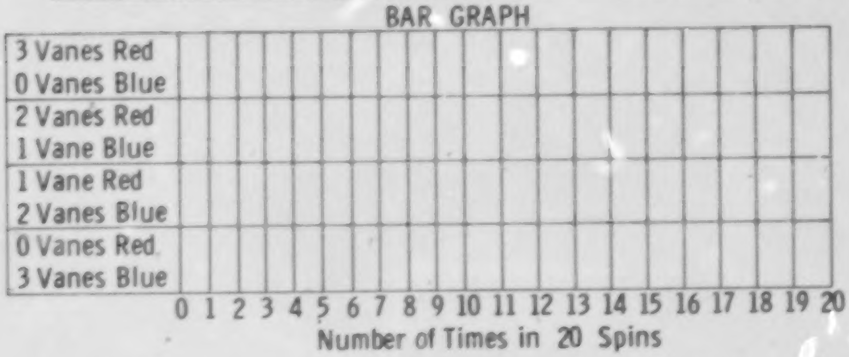
15

4. Add one more vane to the windmill you made so it will have three vanes. Color this third vane red on one side and blue on the opposite side. Blow on the vanes and then place the windmill on a flat surface to see which vanes are up. Mark the chart to show how the vanes stopped and complete the graph from the totals. Compare your results with the Pascal Triangle.

CHART

Spin Number	3 Vanes Red 0 Vanes Blue	2 Vanes Red 1 Vane Blue	1 Vane Red 2 Vanes Blue	0 Vanes Red 3 Vanes Blue
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
Totals				

BAR GRAPH



a. Were your results about what you thought they would be? ________

b. If you did this experiment 900 times, about how many times would you expect all 3 vanes to stop with red up?

[✓ 100] 200 300 400 500

You might like to try this experiment with a windmill of 4 or more vanes.

LESSON 11

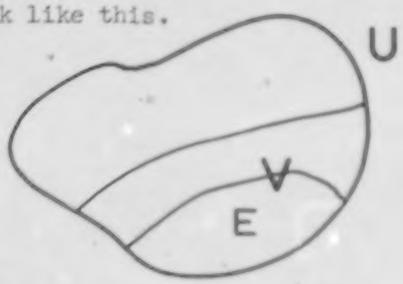
Conditional Probability

Introduction.

In Lesson 9 we drew twice with replacement and later twice without replacement, from a bag which contained a red, a green, and a white marble. Probabilities were then determined. We could very well use these same activities to introduce the idea of conditional probability; for example, what is the probability of green on the second draw when we know that we got red on the first draw? However, to employ a different activity, we will use three tosses of a coin to help children think about conditional probability.

A table of all possible outcomes will enable pupils to count outcomes to determine probabilities as they complete the first two pages in the lesson. The exercises will give further insight into conditional probability.

It might occur to some pupil that every probability is really a conditional probability. This, in a sense, is true except that we choose our language so that the condition is implied or understood. Think of an event as a subset E of a given set U. The set U is the set of "all possible outcomes". We speak of P(E), though we might say, P(E|U). If we add a condition so as to restrict to a subset V of U, with E a proper subset of V, we speak of P(E|V). A diagram might look like this.



Vocabulary: Conditional probability, given.

Materials: None

Suggested Procedure:

State that we have had many opportunities to learn about probability and that now we will find that the probability of an event may depend upon what we know about some condition concerned with it; that is, with our information about it or the circumstances surrounding it.

Suggest that the pupils turn to page 77 in their texts. Complete this and the next page as a class activity. Children should have a fair introduction to conditional probability after they have answered and discussed their answers to the ten questions. If further explanation is needed, you might use a bag with three marbles, as was suggested in the Introduction of this lesson.

Optional Further Activity:

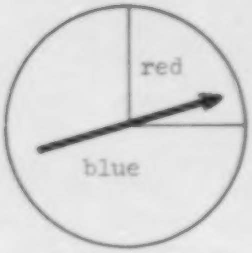
Some children will enjoy this problem. It is written so that it can be duplicated for these pupils.

The Tennis Player

Jack plays tennis every day but Sunday with one of his two best friends and, of course, he likes to win. In fact, if Jack wins a match, he plays against the same friend the next day, but if he loses, he plays with the other friend the next day. When Jack plays Evan, Jack is 3 times as likely to win as Evan is. Jack has 3 chances in 4 to win. When Jack plays Tom, Tom is 3 times as likely to win as Jack is. Jack has 1 chance in 4 to win.

If Jack plays Evan on Monday, is it likely that he will also play Evan on Wednesday? About how many times in one six-week period do you think Jack will play Evan? About how many times do you think Evan will win?

You can use a spinner to see what might happen. You will need a spinner, like this:



You will need a record, like this:

First week	Second week	Third week
Mon.: J E	Mon.: J _	Mon.: J _
Tues. J _	Tues. J _	Tues. J _
Wed. J _	Wed. J _	Wed. J _
Thurs. J _	Thurs. J _	Thurs. J _
Fri. J _	Fri. J _	Fri. J _
Sat. J _	Sat. J _	Sat. J _

Fourth week	Fifth week	Sixth week
Mon. J _	Mon. J _	Mon. J _
Tues. J _	Tues. J _	Tues. J _
Wed. J _	Wed. J _	Wed. J _
Thurs. J _	Thurs. J _	Thurs. J _
Fri. J _	Fri. J _	Fri. J _
Sat. J _	Sat. J _	Sat. J _

When Jack plays Evan, think of the blue part of the spinner as a win for Jack. If the spinner stops on blue, draw a ring around the J on the record, to show that Jack won. Write E on the next line to show he will play Evan again the next day. If the spinner stops on red, draw a ring around the E and write T on the next line to show that Jack will play Tom the next day.

When Jack plays Tom, the spinner must stop on red for Jack to win. If it stops on blue, Jack will lose and he will play Evan the next day. Show what happened in your spinner game here:

Jack played Evan ______ times in 36 days. Jack won ______ of his matches with Evan.

Jack played Tom ______ times in 36 days. Jack won ______ of his matches with Tom.

Pupil pages 79 and 80: After finishing these pages, some pupils might enjoy writing several problems in conditional probability for the others to solve. For example, a die is tossed. What is the probability that a 2 is up, given that the number that is up is even? ($\frac{1}{3}$.) Or, a pupil could write his name, Charles. A letter is picked at random. What is the probability that it is "a", given that a vowel was selected? ($\frac{1}{2}$.) Or, two dice are tossed. The sum of the numbers on the top faces is 10. What is the probability that one of the dice shows a 6 ? ($\frac{2}{3}$.) Boys and girls could think of many interesting examples of conditional probability.

Conditional Probability - Lesson 11.

A coin is tossed 3 times. What is the probability that at least two are heads? Listing the outcomes in a table, we have:

3H	2H	1H	0H
H H H	H H T	H T T	T T T
	H T H	T H T	
	T H H	T T H	

There are a total of 8 outcomes. The outcomes with at least 2 heads are: HHH, HHT, HTH, THH. There are 4 outcomes which have at least two heads. So

P(at least 2 heads) = 1/2.

Now suppose someone whispered to you, "The first toss was a head." What is the probability of 2 heads now? Look at the table. The outcomes with a head on the first toss are: HHH, HHT, HTH, HTT. How many outcomes have a head on the first toss? 4

Of these, how many have at least two heads? 3

P(at least 2 heads if you know the first toss is a head) = 3/4.

This could be abbreviated to

P(at least 2H | H on first toss) = 3/4.

This is read, "The probability of at least two heads given a head on the first toss." In this example, the probability of 2 heads increased from 1/2 to 3/4 when we knew that the first toss was a head.

What is the probability of at least 2 heads if we know that the first toss shows a tail? A list of the outcomes with a tail on the first toss is:

THH, THT, TTH, TTT

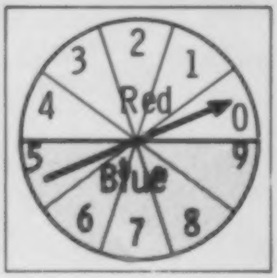
Of these, how many had at least 2 heads? 1

P(at least 2H | T on first toss) = 1/4.

Continue to use the table of outcomes for 3 coins to answer the following questions.

1. P(at least 1H | T on first toss) = 3/4.
2. P(3T) = 1/8.
3. P(3T | T on first toss) = 1/4.
4. P(at least 1H) = 7/8.
5. P(at least 2H | 2H on first two tosses) = 1.
6. P(at least 1H | T on second toss) = 3/4.
7. P(exactly 2 tails) = 3/8.
8. P(exactly 2T | T on second toss) = 1/2.
9. P(exactly 2H | H on first toss) = 1/2.
10. P(at least 1H | exactly 1 tail) = 1.

Exercises - Lesson 11.



Use this spinner. The dial is divided into 10 equal regions.

1. On one spin

a. P(Blue) = 1/2.

b. P(Red) = 1/2.

c. P(2) = 1/10.

d. P(Not 2) = 9/10.

e. P(7) = 1/10.

f. P(Not 7) = 9/10.

g. P(even) = 1/2.

h. P(odd) = 1/2.

2. The 10 possible outcomes are:

R0, R1, R2, R3, R4, B5, B6, B7, B8, B9

a. P(both red and even) = 3/10. To answer (a.), find those outcomes which are red (R0, R1, R2, R3, R4), and of those which are even (R0, R2, R4). Thus 3 of the 10 possible outcomes are both red and even.

b. P(both blue and even) = 1/5.

c. P(red | even) = 3/5. Remember that this means, "The probability of red given that it is even."

d. P(red | odd) = 2/5.

3. Use the same spinner. List the numbers which are multiples of 4.

0, 4, 8 (be sure that you list three)

a. P(multiple of 4) = 3/10.

b. P(both red and a multiple of 4) = 1/5.

c. P(either red or a multiple of 4) = 3/5.

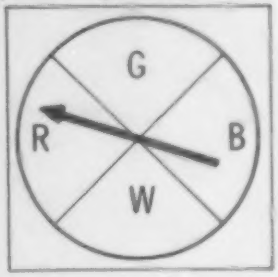
d. P(both even and a multiple of 4) = 3/10.

e. P(red|multiple of 4) = 2/3. f. P(multiple of 4) = 3/10.

g. P(multiple of 4|red) = 2/5. h. P(multiple of 4|blue) = 1/5.

i. Which has the greatest probability, (1), (2), or (3) ? 2
  (1) P(even number)
  (2) P(even number|red)
  (3) P(even number|blue)

4. Complete this table of outcomes of two spins on this spinner. It has four equal regions.



		Second Spin			
		R	W	B	G
First Spin	R	RR	RW	RB	RG
	W	WR	WW	WB	WG
	B	BR	BW	BB	BG
	G	GR	GW	GB	GG

On two spins, the:

a. Total number of outcomes = 16.

b. P(RR) = 1/16.

c. P(at least one R) = 7/16.

d. P(both R on first and G on second) = 1/16.

e. P(R on first|G on second) = 1/4.

f. P(R on second|G on first) = 1/4.

g. P(at least one R | R on first) = 1.

h. P(2R | R on first) = 1/4.

i. P(at least one W | R on second) = 1/4.

j. Which has the smallest probability, (1), (2), or (3) ? 3
  (1) P(WW | W on first)
  (2) P(WW | W on second)
  (3) P(WW)

LESSON 12

Ghosts, Goblins, and "Coins that Remember"

Introduction.

This lesson should help pupils to realize that the law of averages is often misused or misunderstood. It is hoped that in the future they will recognize and avoid the errors that are commonly made in its name.

The wide misuse of this so-called "law" is ample evidence that the basic concepts of probability are not known to many otherwise well-informed people. In newspapers, magazines, and other mass media of communication, reference is made to the law of averages and many times the judgments and conclusions which result are *just plain incorrect*!

If this brief introduction to probability does nothing more than cause the pupil to question the use of the law of averages, it has served an important purpose.

Vocabulary: Conclusions, average.

Materials: None.

Suggested Procedure:

This lesson might best be taught by first letting the pupils read the pupil pages 80 through 85 and then discussing these in class. You may want to use the Pascal triangle in the discussion of Statement A, raising such questions as:

> If we toss 5 coins, what is the probability
> that all 5 will be heads? $(\frac{1}{32}.)$

The discussion of statements about the law of averages which pupils have heard should be most interesting. Try to collect some of these references from newspapers and magazines to share with the class.

Pupils may question that some of the statements on page 84 are false. They may enjoy searching for information relating to these statements. The book, *Believe It or Not*, by Robert L. Ripley, might appeal to many pupils.

The experiment to be done at home can be done at school if you have time for it. It should help pupils see that an "average" may conceal a wide range of variations. Call attention to the large number of tosses that are sometimes required to obtain a head while at other times heads are tossed several times in a row. Notice that we would expect the pattern heads-tails-heads-tails-heads-tails-heads-tails-heads-tails only once in every 1,024 times.

Pupil page 86: These exercises are more suitable for class discussion than for independent work.

Lesson 12 - Ghosts, Goblins and "Coins that Remember".

1. Do you believe that there are ghosts?
2. Do you believe that there are goblins?
3. Do you believe that a coin can remember?

You probably answered "No" to all these questions. Yet often we hear people talk as if they believe that coins can think and remember. They really do not understand the ideas in the *law of large numbers*. Most people call it the *law of averages*, and they often draw wrong conclusions from it.

You have heard people say:

A. "I have tossed an honest coin four times. Each time it came up heads. The law of averages says that the next toss will be tails."

Do you believe that the next toss is more likely to be tails than heads?

B. "My teacher uses a spinner to assign positions for the baseball game of "work up". I haven't been assigned as a pitcher yet this year. Therefore, by the law of averages, I'm sure to be assigned as pitcher today."

Do you think that this pupil is more likely than not to be chosen as a pitcher?

C. "I have been tossing an honest die. In 23 tosses, the face with one dot on it has never been up. By the law of averages, it is very likely that it will come up on the next toss."

Do you think the face with one dot is more likely than any other face?

Let's look at each of these examples of a misunderstanding of the "law of averages". Look back at statement A.

> A. A coin does not have a memory. It cannot "remember" that it has been heads on the last four tosses. There is an equal chance for heads or for tails on the next toss.

We can use mathematics to prove that it is "unusual" to have a coin show four heads in four tosses. We can draw a tree diagram, make a table, or look at the fourth row in the Pascal Triangle. How many different outcomes are there when 4 coins are tossed or when one coin is tossed 4 times? How many of these outcomes consist of four heads? So, $P(4\text{ heads}) = \frac{1}{16}$.

However, this also means that we expect 4 heads in a row, once every 16 times that we toss 4 coins. The coin while flying through the air on the fifth toss cannot say to itself, "Well, that's 4 heads in a row; I better twist a bit more and be sure to land tails or I'll mess up the 'law of averages'." The probability of heads on the next toss is of course $\frac{1}{2}$, the same as any other individual toss. Some people who misunderstand the law of averages think the probability of tails is much greater than $\frac{1}{2}$ after a coin has been heads several times in a row. Do you know people like this? They have forgotten that what happens on one toss has NOTHING to do with what will happen on the next toss.

Refer to statements B and C.

B. If there are 9 positions on the baseball field, then the probability of getting any one position is 1 out of 9. The fact that this pupil has not been a pitcher yet does not cause the spinner to favor one position for him over the others. He still has only 1 chance in 9 of being a pitcher today.

C. This person is overlooking one simple fact about a die -- it cannot think! It cannot say, "Let's see now. I know the probability of any face is $\frac{1}{6}$. My face with one dot on it has not been up in 23 tosses, so on the next toss I'll land so that the face with the one dot is on the top."

This person is thinking, "One face hasn't been up for a long time, so that face is more likely to come up than any of the others." This is a mistake about the law of averages that people often make. He doesn't really believe that dice can think, yet he is acting as if they could. Each face on a die has just as much chance to be up as any other face. If the face with one dot has not been up in 100 tosses, it still has no more chance than any other face to be up on the next toss. In fact, it has just one chance out of six.

Can you think of other correct or incorrect statements that you have heard about the law of averages? List some of them.

____

____

____

Why do so many people misunderstand the law of averages? It is too bad, but we all believe things and arrive at conclusions which just aren't true.

Which of these statements are false?

1. Lightning never strikes twice in the same place.
2. If you handle a frog, you'll get warts.
3. The end of the Panama Canal on the Pacific Ocean side is farther west than the end on the Atlantic Ocean side.
4. Horses are smarter than pigs.
5. George Washington threw a dollar across the Potomac River.
6. Columbus discovered America.

Many people believe some of these statements. Did you believe any of them? If you did, it isn't at all surprising. However, the six statements are all false. Most of us believe some things which really aren't true. Why is this so? There are many reasons. Among them are:

1. We are told or we have read something which is not true, but we remember it.
2. We did not understand what we were told or what we have read.
3. We reasoned incorrectly.
4. Our experience caused us to believe something that wasn't true.

5. We jumped to a conclusion without knowing enough facts.
6. We failed to check our belief against the facts.

This list could go on and on. There may be other reasons that you can think of. People arrive at false ideas about the law of averages for many of these same reasons. We can be fooled unless we are very careful. We might believe that an outcome, such as heads on a toss of a coin, is bound to happen if it hasn't happened for many tosses. It is easy to understand how our brain fools us in this case. It tells us that for a large number of tosses of a coin, heads will occur about half of the time -- and this is true. This is an example of the law of large numbers. Then we observe that heads hasn't occurred for several tosses and we make the mistake of thinking that heads must now start occurring more often to "catch up" with the number of tails. This is not true. Remember, a coin can't think. On each toss, there is just as much chance for heads to turn up as for tails.

By using mathematics, we can learn many interesting things. For example, from 15 children in your room, there are 6,435 different ways you can have 7 children on a committee. If you choose a 7-member committee from 30 students, you have a choice of 2,035,800 different committees. Another example is if a coin has been tossed and heads have occurred 7 out of 10 times, chances are less than $\frac{1}{2}$ that tails will "catch up" in 100 tosses. The mathematician can tell what will probably happen in cases such as this.

The next time that you hear some statement about the "law of averages", listen carefully. Try to find what the person believes and see if he is using it correctly.

Exercises - Lesson 12.

Mark these TRUE or FALSE.

False 1. You have been spinning a spinner that has a dial which is $\frac{1}{2}$ black and $\frac{1}{2}$ red. The last four spins have landed on black. It is more likely that the spinner will show red on the next spin than black.

False 2. The last five new pupils who came to our school were boys. The chances are better than equal that the next new pupil will be a girl.

False 3. The hospital reported that the last seven babies born there were girls. It is more likely that the next baby born there will be a boy than that it will be a girl.

False 4. The weatherman says that on the average it rains 4 days during the month of July. Today is the 27th of July and it has not rained all month. Therefore, it will rain tomorrow.

False 5. An auto dealer has 250 new cars and he knows that one out of every five new cars he sells is colored black. This week he has sold a blue, a white, a green, and a grey car. It is more likely than not that the next car he sells will be a black one.

Things To Do At Home - Lesson 12.

This experiment may help you to see why some people draw wrong conclusions from the law of averages.

Problem:

How many times, on the average, do you think that you would have to toss a coin before it comes up heads? Use the chart on the next page.

Procedure:

Toss a coin. Count the number of tosses until you get a head. For example: If you get a head on the first toss, write 1 in the column just to the right of "1st head". Start over. If you do not get a head until the fourth toss, write a 4 just to the right of "2nd head". Continue until you have completed column A. Repeat for columns B through E. Each column provides spaces to record the tosses for 10 heads.

After you have tossed 50 heads, add the number of tosses to get each group of ten heads. Divide each of these sums by 10 to find the average number of tosses needed to get one head.

Then, add the sums from the five columns and divide by 50. This gives the average number of tosses to get one head. Is this average closer to 2 than the average for each of the five columns? How many times did it take more than 5 tosses to get a head? How many times did it take 2 tosses to get a head? How many times did it take only 1 toss to get a head?

Number of Tosses to Get a Head

	A	B	C	D	E
1st head					
2nd head					
3rd head					
4th head					
5th head					
6th head					
7th head					
8th head					
9th head					
10th head					
SUM					
SUM ÷ 10					

Perhaps now you can see why some people misunderstand the law of averages. With many tosses of a coin, we do see that about half of the tosses are heads. That is, it takes 2 tosses, on the average, to get heads. But, when you tossed a coin, you found that sometimes you tossed a head on only 1 toss. Other times, you had to toss the coin several times to get a head. This should help you understand that these people fail to see that the "average" is made from numbers that differ quite widely and that there is NOT a "law" which says that you must get a head after tossing 5 tails, for example.

APPENDIX - Probability Devices

This appendix suggests devices which you might like to make. You can think of games which can be played with them. Decide how to organize and record the data. These devices can help you in your study of probability.

1. Regular Solids

We have used coins, dice, cubes, cards, and other materials to help us learn about probability, but there are many devices which are just as useful. The patterns on the next six pages are for the construction of regular solids which can be used in probability experiments. The patterns can be traced on tagboard and the solids will then be sturdy enough to toss or roll.

The tetrahedron, octahedron, and hexahedron (cube) are not difficult to construct. Just fold on the dotted lines and glue the tabs.

The dodecahedron is more difficult to construct. Make the first half of it by cutting to the dotted line at each arrow. Then fold on the dotted lines and glue the tabs. Complete by folding the second half and gluing it to the first half, tab by tab. Do not make both halves and then try to put them together.

The icosahedron appears difficult to construct, but it isn't. Cut to the dotted line at each arrow. Then fold and glue the tabs in order, one by one, and it will work out nicely.

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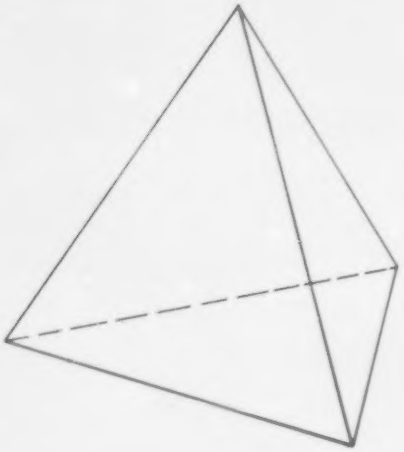
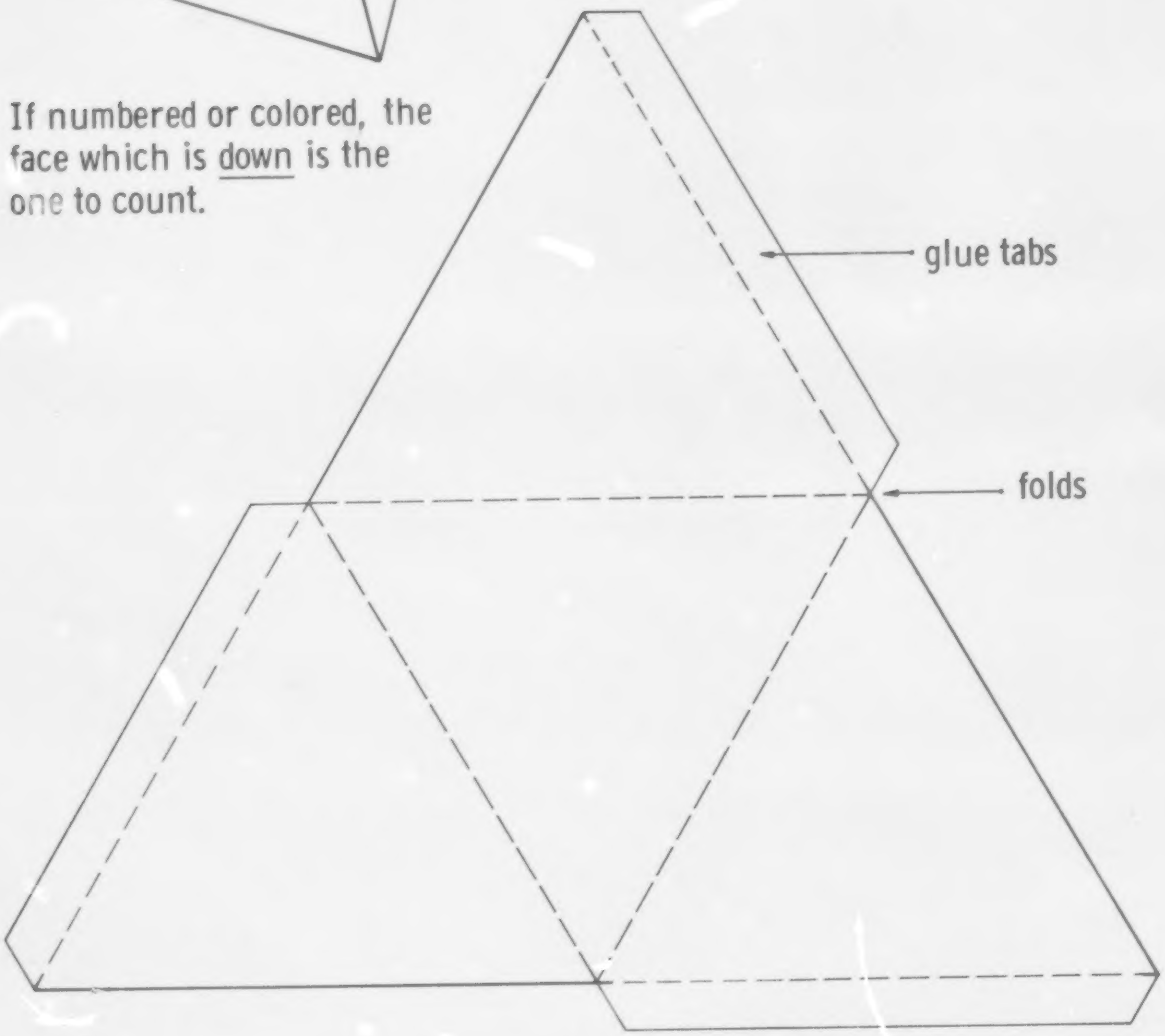
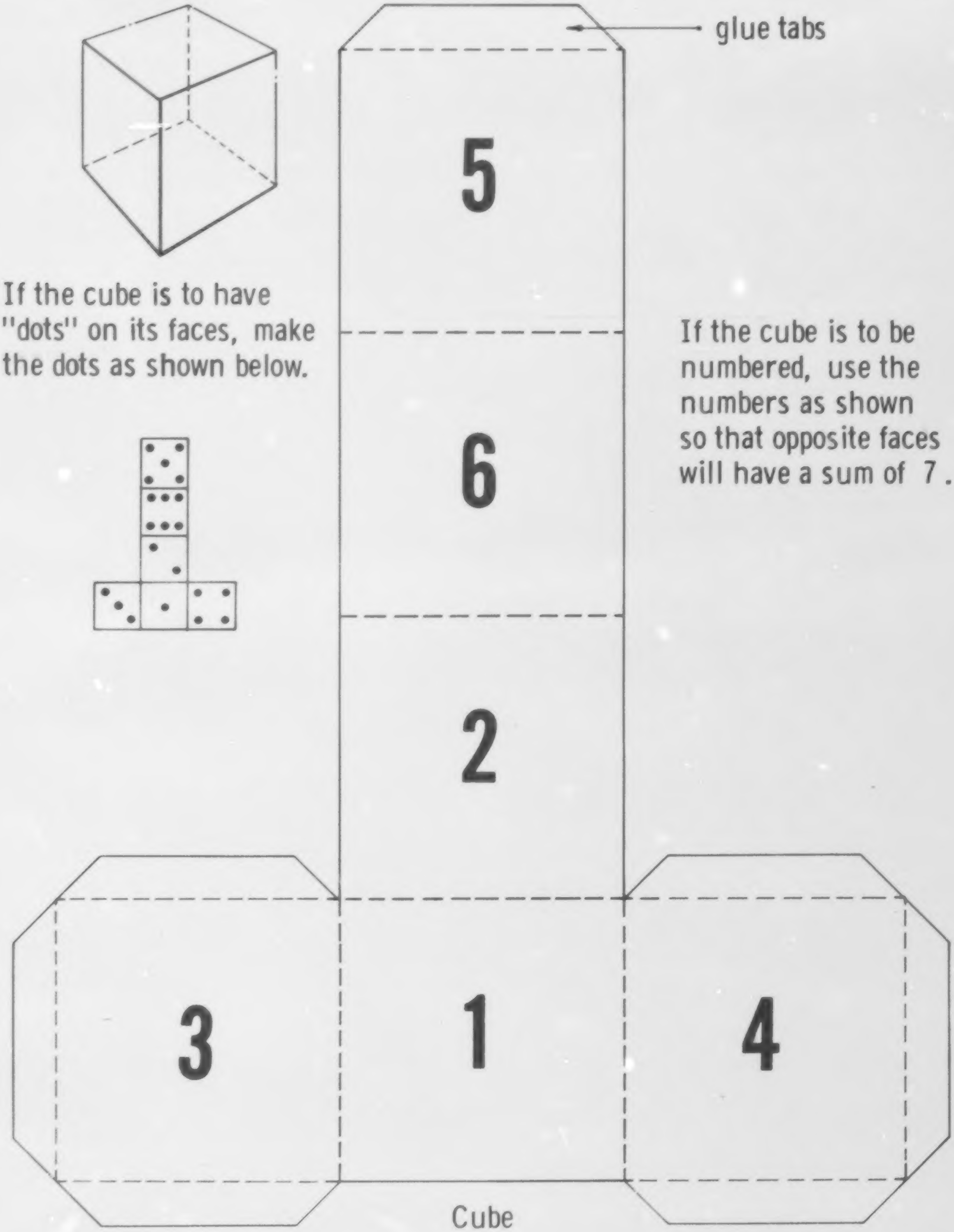


If numbered or colored, the face which is down is the one to count.


glue tabs
folds

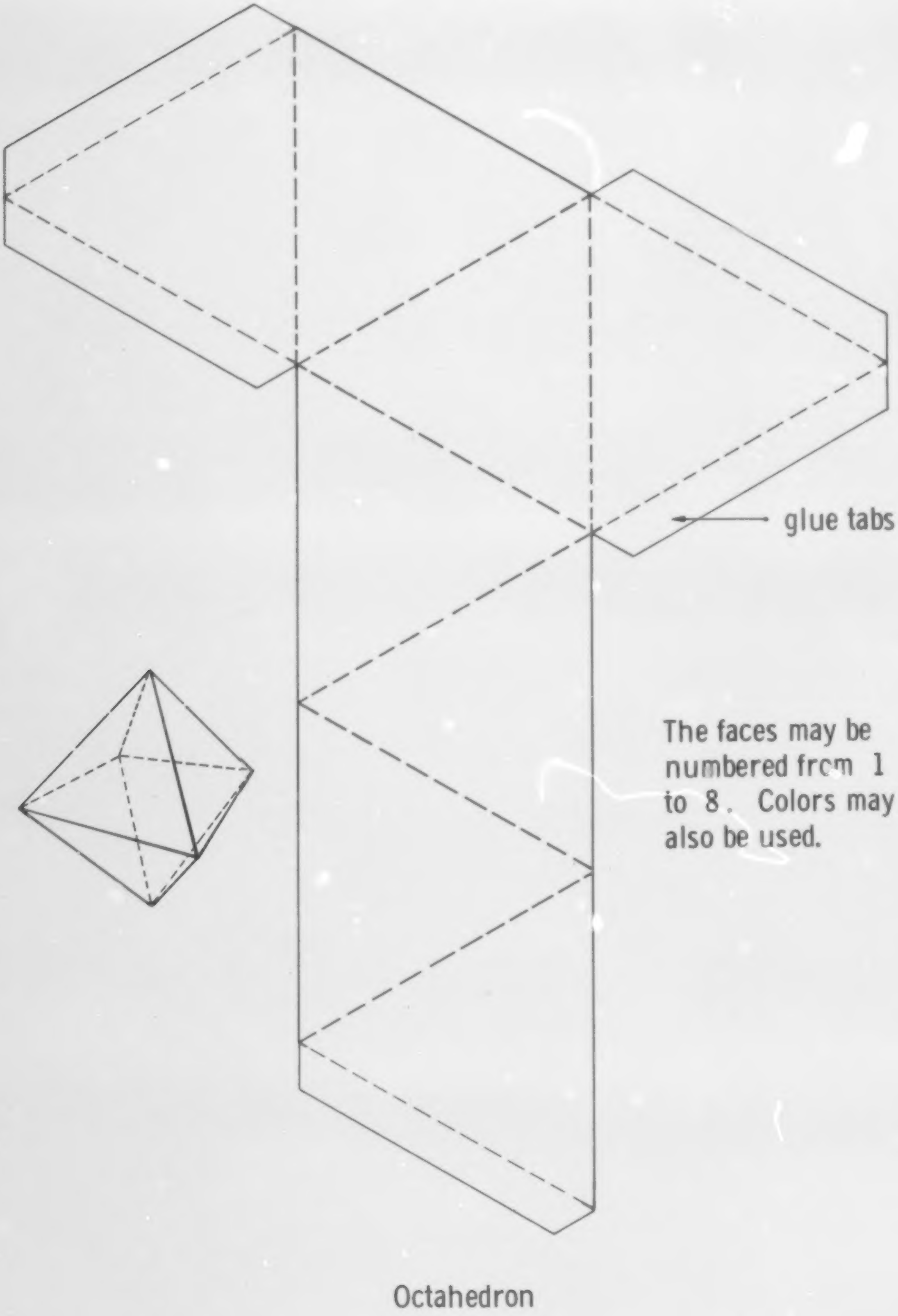
Tetrahedron



If the cube is to have "dots" on its faces, make the dots as shown below.

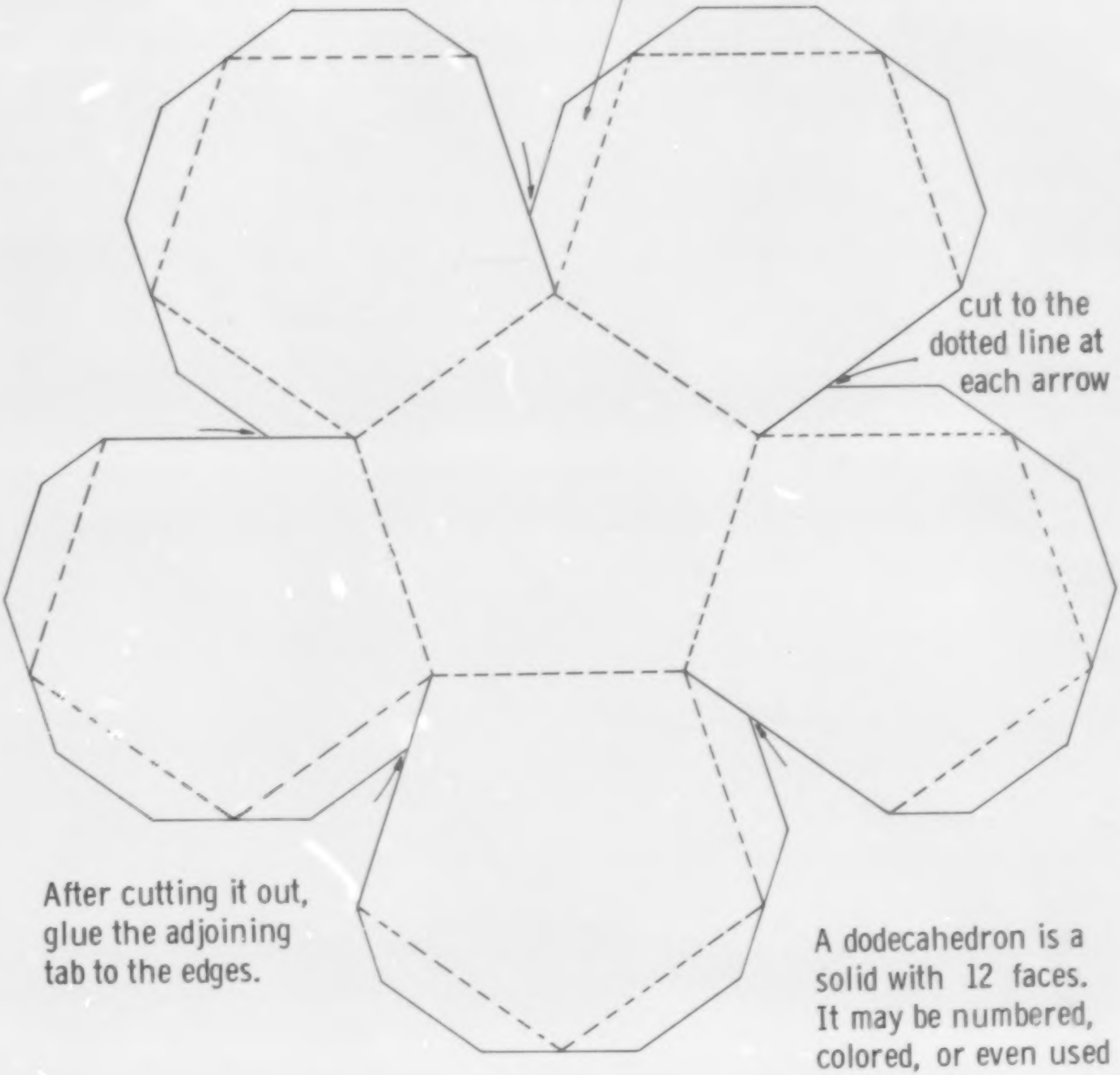
If the cube is to be numbered, use the numbers as shown so that opposite faces will have a sum of 7.

Cube
or
Hexahedron



The faces may be numbered from 1 to 8. Colors may also be used.

Octahedron



glue the tabs

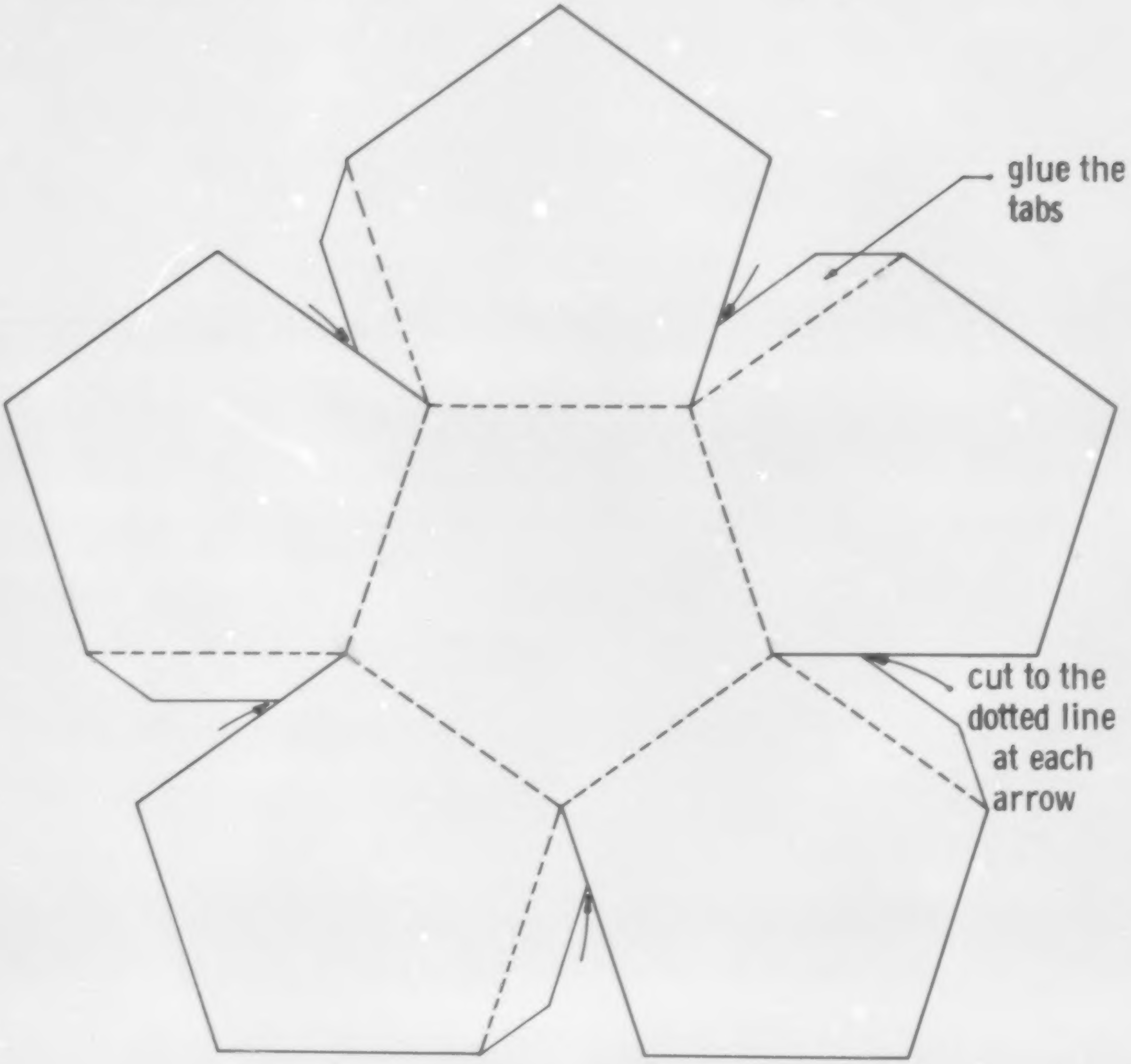
cut to the dotted line at each arrow

After cutting it out, glue the adjoining tab to the edges.

A dodecahedron is a solid with 12 faces. It may be numbered, colored, or even used as a calendar

First half of a Dodecahedron

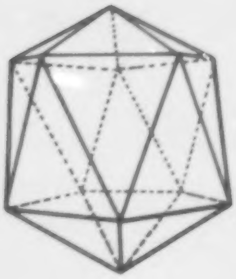
After cutting this out, do not
glue adjoining tabs to the edge.
Instead, glue one tab at a time
to the first half of the dodecahedron.

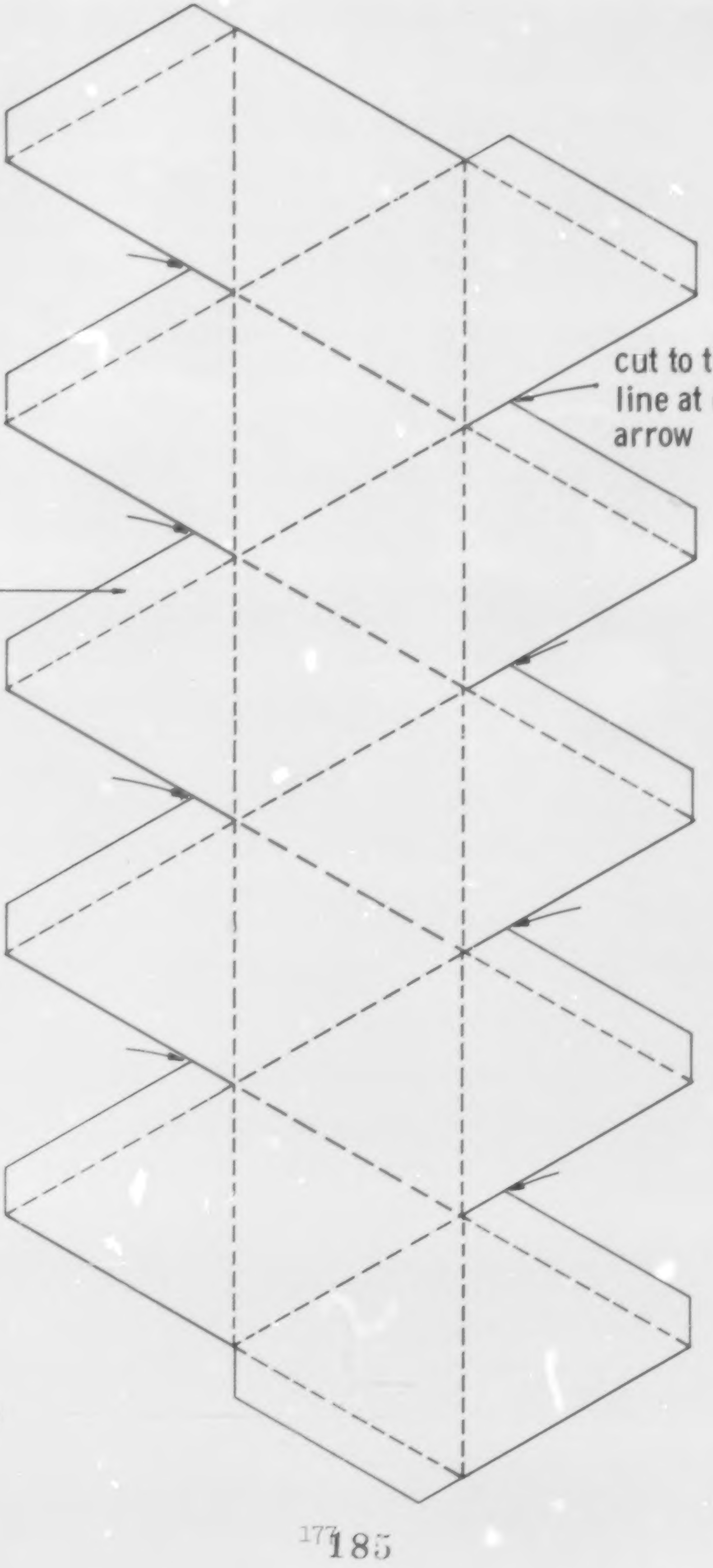


glue the tabs

cut to the dotted line at each arrow

Second half of a Dodecahedron




cut to the dotted line at each arrow
glue the tabs

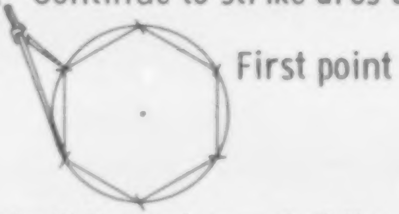
A solid with 20 faces

Icosahedron

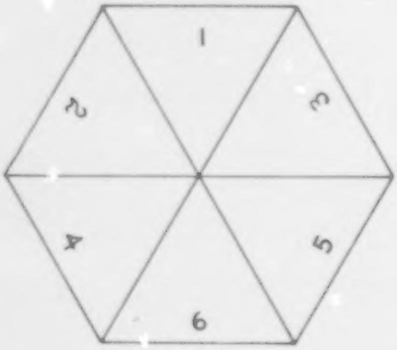
2. Tops

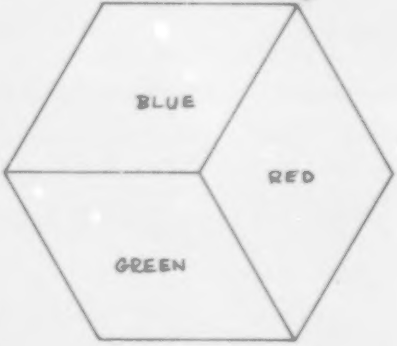
For a small top you need a piece of cardboard, a wood dowel or sucker stick, a straight edge, and a compass. Use a two-inch piece of dowel or sucker stick. Sharpen one end to a point with a pencil sharpener.

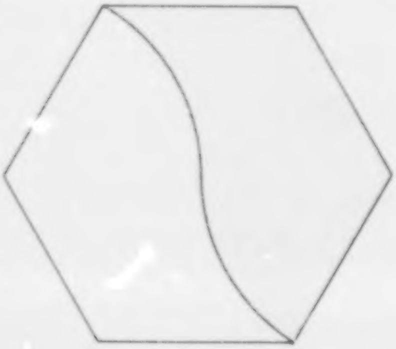
Use a compass and straight edge to make the dial from cardboard. Mark a point for the center and draw a circle with radius $1\frac{3}{4}$ inches. Mark any point on the circle and, with that point as center, strike an arc with the same radius to intersect the circle. Continue to strike arcs around the circle.

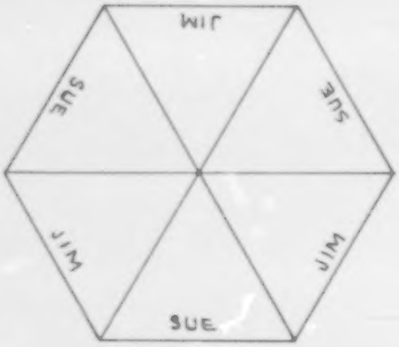

First point

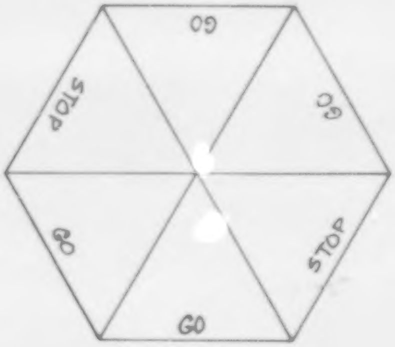
Connect points marked and cut off the outer part of the region to make a hexagonal region. Divide it as desired. Here are some suggestions:




BLUE
RED
GREEN



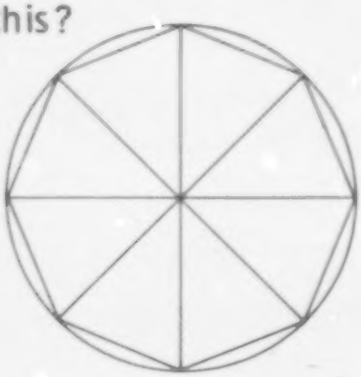

JIM
SUE
SUE
JIM
JIM
SUE

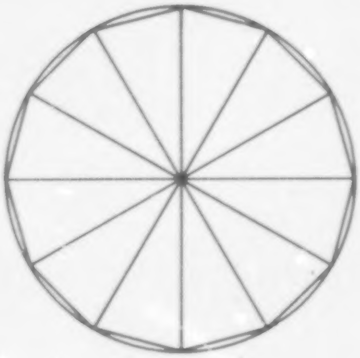

GO
STOP
GO
GO
STOP
GO

Poke the sharpened end of the stick through the center of the hexagon. (A drop of glue will help to keep it from wearing too large a hole, but experiment first to find the best balance for the top.) Spin the stick between thumb and forefinger. Spin the top on a flat surface. The edge that stops against the table is the one that is counted.

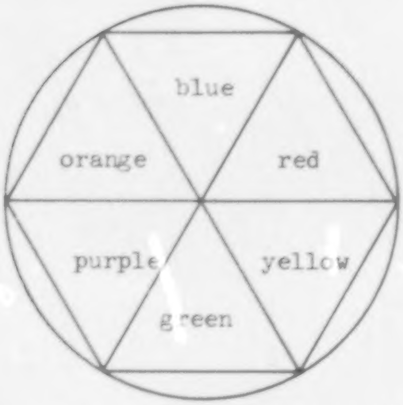
A large top may be made in the same way. An ordinary pencil makes an adequate stick. It is a good idea, however, to slit the cardboard along the division lines at the center before inserting the pencil. Use glue to fasten the cardboard to the pencil.

The top can also be made with eight edges or with twelve. Do you know how to do this?



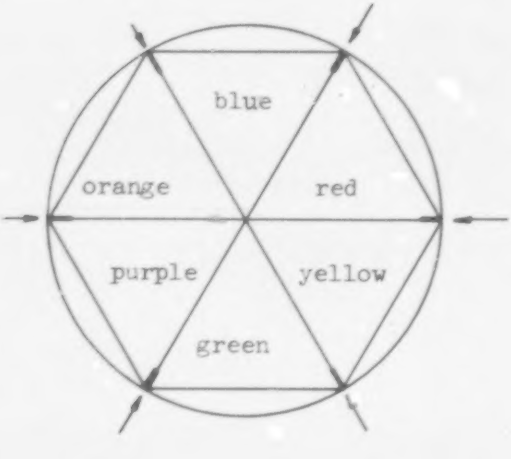


You can make a top with two or three dials. Use a sucker stick or thin dowel $3\frac{1}{2}$ inches long and sharpened at one end. Make a circular cardboard dial 6 inches in diameter. Inscribe a hexagon, but do not cut it out. Color as shown:



Insert the stick through the center and glue the dial to the stick.

When the glue is dry, place over the dial a "bearing" made of a piece of milk carton 1 inch square with a hole in the middle. Make another dial 6 inches in diameter, but if possible use cardboard that is slightly lighter in weight. Inscribe a hexagon, color, and cut as shown:

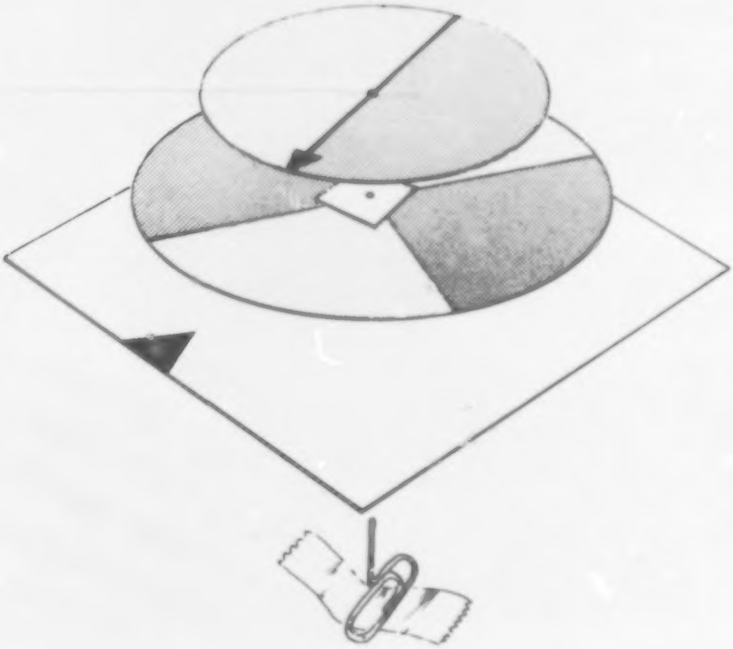


Make cuts $\frac{5}{8}$ inches long on the lines, as shown by arrows. Bend the cardboard up on the right of each cut to make a triangular "wind-catcher".

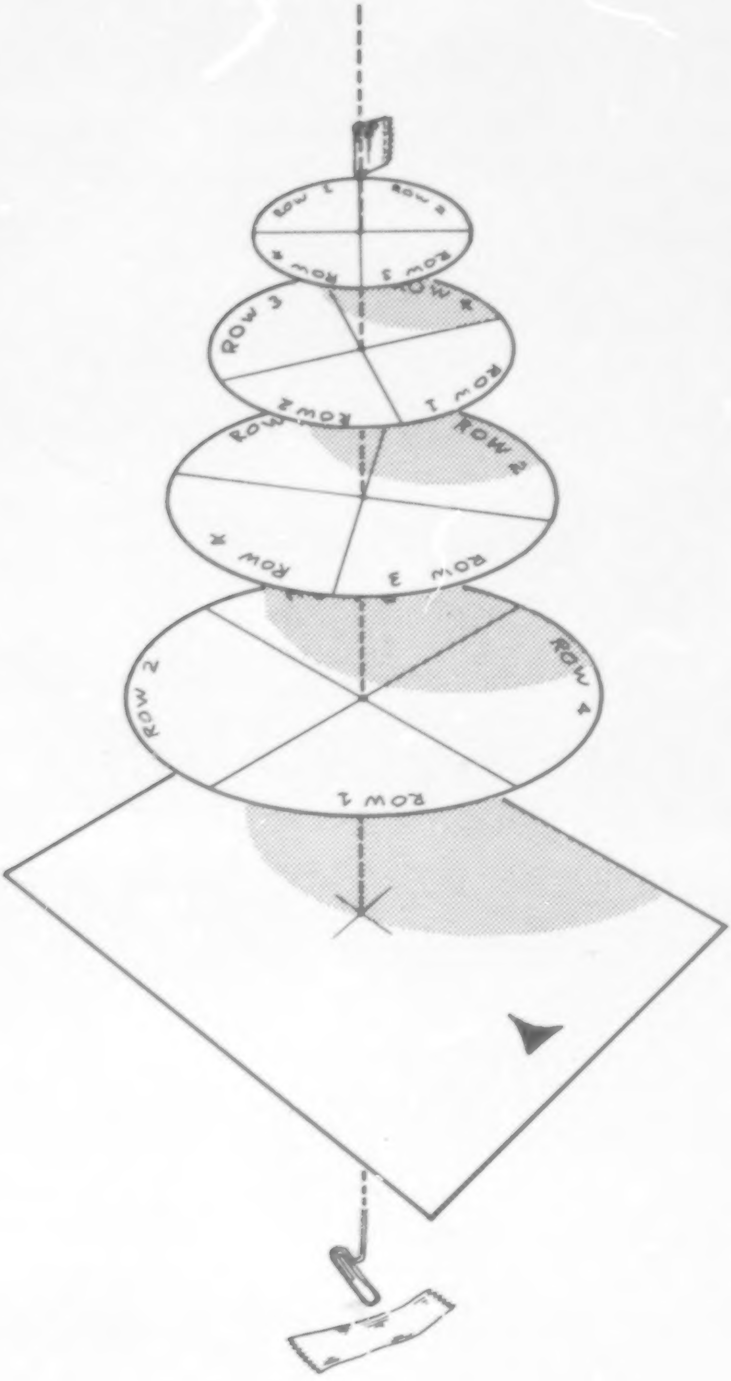
Place this dial on top of the bearing. Before spinning the top, line up the two dials so that the colors match. The bent-up cardboard will let you do this. Spin. Record the number of spins necessary before the colors again match. How many spins would you expect it to take?

3. Spinners

Spinners which come with most games have a fixed dial and an arrow which spins. You can make this type of spinner. Another spinner which is easily constructed is one in which the dial spins, as shown in the drawing. Construct the base out of heavy tag board. Use a heavy paper clip for the post and then various dials may be exchanged and used on the base.

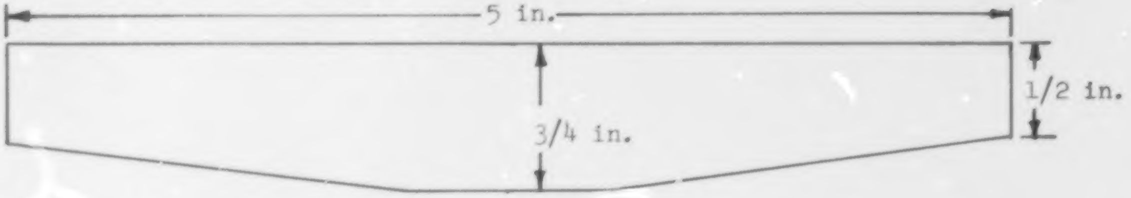


Another spinner with dials which go from large to small can illustrate how items can be put into an order. Each heavy cardboard dial is divided into a number of equal parts. The number of parts is the same as the number of items to be ordered. For example, to experiment to see how four rows in a classroom might be dismissed for lunch, each of four dials is divided into four equal parts. (A dial is made for each row.) Spin the dials and record how they line up with the arrow on the base. This device can be used to illustrate an orderly way of arriving at and listing the various arrangements.


ROW 1
ROW 2
ROW 3
ROW 4
ROW 3
ROW 4
ROW 1
ROW 2
ROW
ROW 2
ROW 4
ROW 3
ROW 2
ROW 4
ROW 1

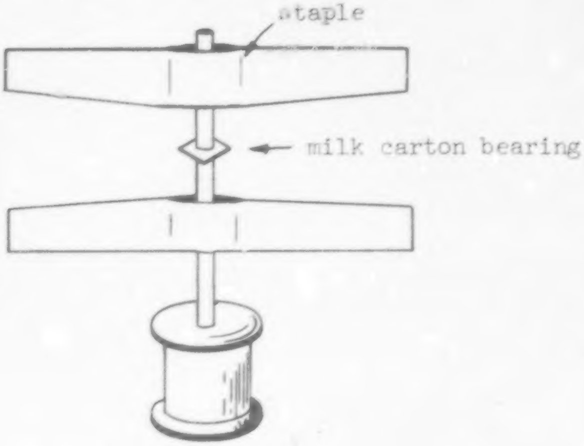
4. A Windmill

Cut four pieces of file card or tagboard to make two vanes.



A

Staple two pieces together in two places as shown in sketch B. Color one side of this vane red with a pencil or crayon. Insert a sucker stick in the middle between the two pieces. Use a punch to make a hole in a piece of milk carton one-half inch square for a "bearing". Staple the other two pieces of file card together to make a second vane. Mark one side as before, and insert the top of the stick between them. Paste a strip of gummed paper or tape over one end of a small spool. Put the end of the stick into the hole at the other end of the spool. Hold the spool and blow the file card vanes. They should turn quickly and independently.



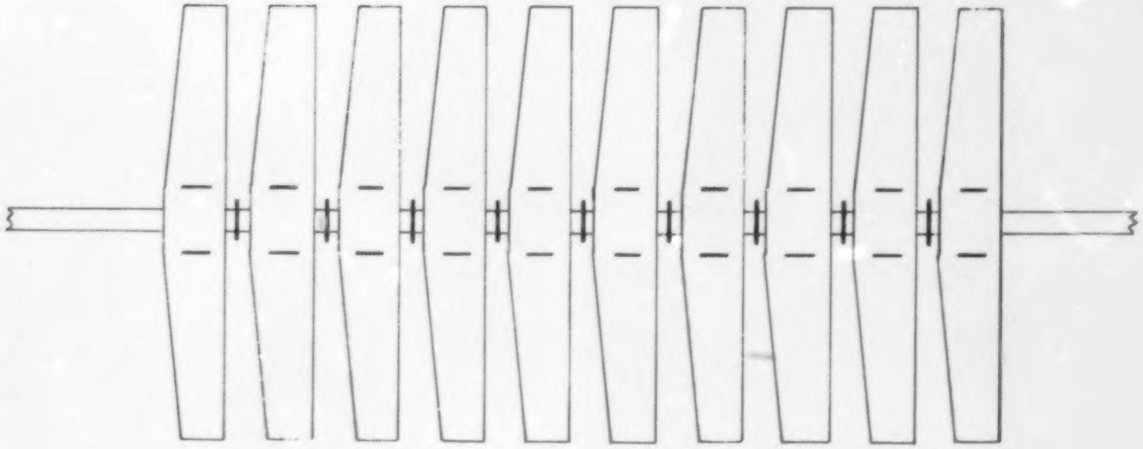
B

Practice blowing a few times. When vanes have stopped turning, lay them gently on a flat surface so that both vanes are flat on the surface. Then record whether both vanes are red, one is red and one is white, or both are white. After fifty trials, do the the results of the experiment fit with the expected results?

Whirly-bird

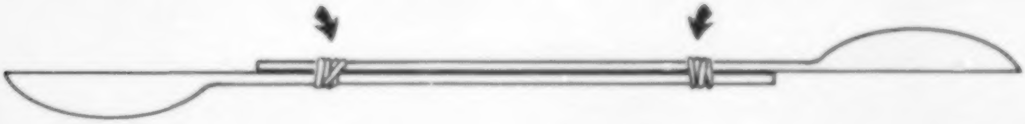
This is another type of windmill.

Use a section of quarter-inch dowel about two feet long. Wax it by rubbing it with a piece of crayon. Follow the directions as given for a windmill and make as many file-card vanes and milk-carton bearings as desired. Hold the dowel at both ends, like a harmonica, and blow. When vanes have stopped turning, lay the Whirly-bird gently on a flat surface so that all vanes are on this surface. One vane corresponds to one coin, so ten vanes can be used to duplicate an experiment of "tossing ten coins". Can you think of other ways to color the vanes so that other experiments can be done?



5. Spoon device

Use two plastic picnic spoons of different colors. Lay the handle of one on the handle of the other so that the bowls are at opposite ends and face opposite ways. Fasten with two rubber bands as shown at arrows.

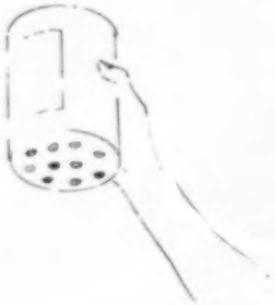


Roll the handles between your palms and drop on a table from a height of a foot or so. The spoon on top counts. Is it just as likely that one spoon will be up as the other?

6. Sampling Boxes (Urns)

Many probability experiments require a sampling to be taken in a random manner. This device uses various colored marbles.

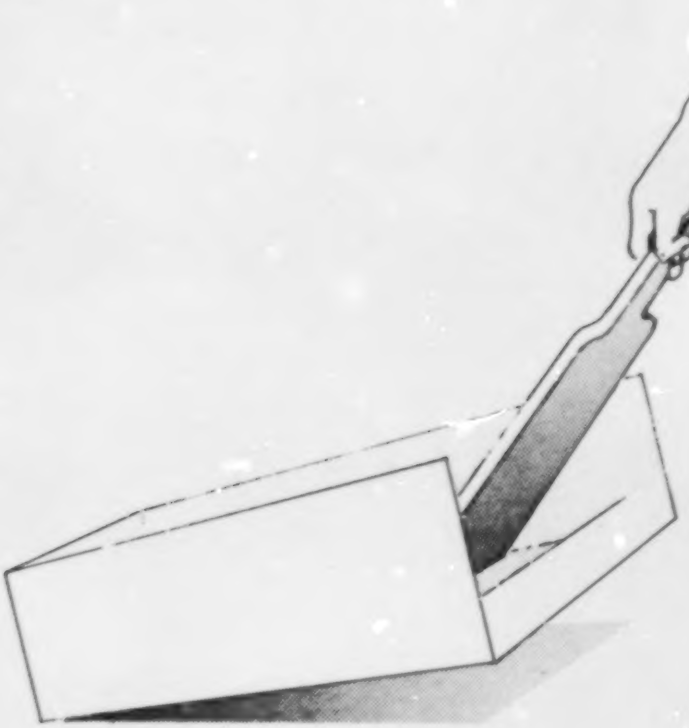
An oatmeal box serves well as the container (urn). Cut round holes, ten, for example, in the bottom of the box. These holes should have a diameter slightly smaller than the marbles so the marbles can be seen in them. Chinese Checker marbles serve well and come in packages of 6 colors, 10 of each.

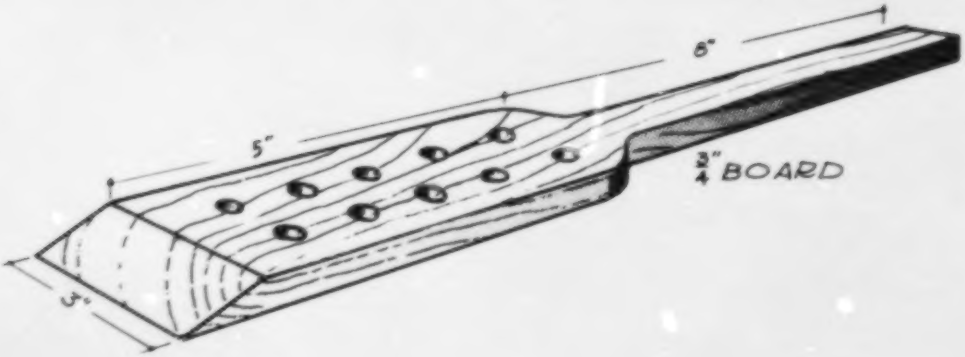


Example: Tell a friend you have 60 marbles in the container. Do not tell him how many there are of each color. Use, for example, 40 white and 20 black. Turn the urn up 10 times and record the number of black and white which show in each sampling. From the total, predict the probable number of white and black marbles in the urn.

A second sampling box can be made by using a box (rectangular or cylindrical) with a lid. It should be large enough to allow objects such as marbles or small balls to roll around. In a corner of a rectangular box or at the bottom edge of a cylindrical one, cut a hole just large enough to let one object go through easily. With the hole up, shake the box. Turn it over in your hand to let one object come out.

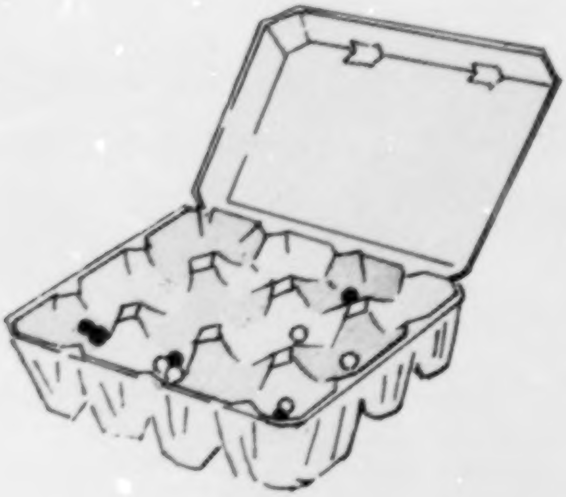
A third sampling device, a paddle, is made as shown in the drawing. The holes are slightly smaller in diameter than the marbles. Cut a box as shown (a shoe box works well). Place marbles in the box. Scoop with the paddle until all "holes" are filled. This gives a sample of the entire collection of marbles.




8"
5"
$\frac{3}{4}$" BOARD
3"

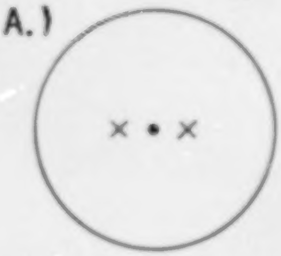
Egg-Carton Sampler

An egg carton and some marbles can be used for experiments. For example, color alternate pockets of the carton black. Place 5 black marbles and 5 white ones inside the carton. Close the lid, turn the carton upside down, and allow the marbles to roll around. Flip the carton upright and open the lid. Record the information you are interested in, for example, the number of black marbles in black pockets, etc.



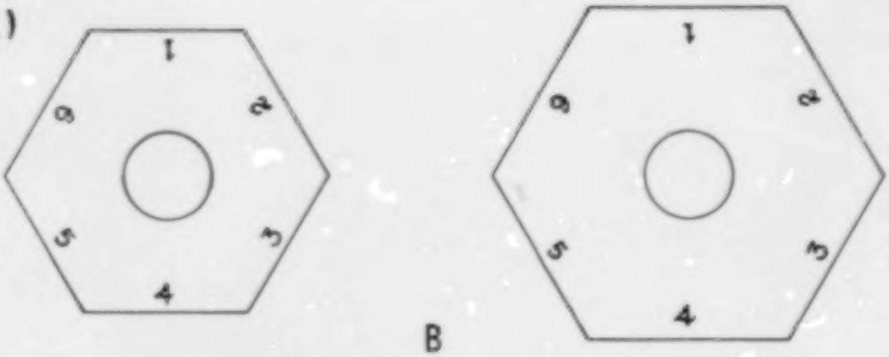
8. Hexawhirl

This gadget works like an old-fashioned button-on-a-string toy. Cut from cardboard two circles with radius $1\frac{3}{4}$ inches. In each, punch two holes just big enough to insert a piece of strong string. The holes should be punched on one of the diameters of the circle, each $\frac{1}{2}$ inch from the center. (See points marked X in Figure A.)



A

Make two or more cardboard hexagons using a $2\frac{1}{2}$ inch radius for the first, a 3 inch radius for the second, etc. Cut a hole with radius $1\frac{1}{4}$ inches in the middle of each hexagon and number the sides 1 through 6. (See Figure B.)



B

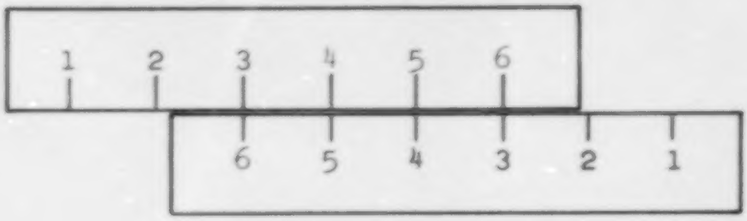
Use a strong string 50 inches long. Insert one end through a hole in one of the circles, through the smaller hexagon, then the larger hexagon (making sure the numbered sides face the same way), and then through the other circle. Leave a loop of string beyond the circle and insert the string through the other hole of the second circle, back through the larger hexagon, the smaller hexagon, and the first circle. Tie the ends of the string together to make a second loop. Adjust the cardboard pieces so that the loops on each

7. A Probability Scale

When tossing two dice or other regular solids, it helps to construct a table for counting how many ways a certain sum or product can be obtained. Two cubes, for example, give the following sums.

Sums	Possible Combinations	No. of Ways
2	(1,1)	1
3	(1,2), (2,1)	2
4	(1,3), (2,2), (3,1)	3
5	(1,4), (2,3), (3,2), (4,1)	4
6	(1,5), (2,4), (3,3), (4,2), (5,1)	5
7	(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)	6
8	(2,6), (3,5), (4,4), (5,3), (6,2)	5
9	(3,6), (4,5), (5,4), (6,3)	4
10	(4,6), (5,5), (6,4)	3
11	(5,6), (6,5)	2
12	(6,6)	1
	Total	36

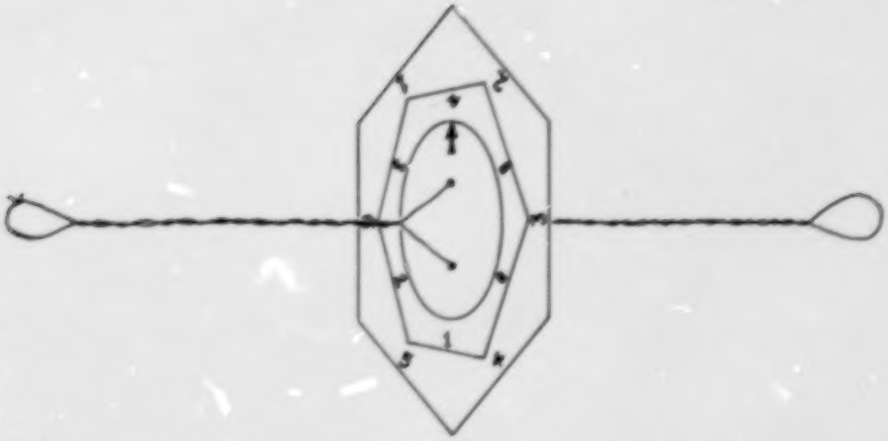
This information can be placed on two number lines on strips of cardboard as in the sketch.



These two strips can be placed in pockets of a larger piece of cardboard so that by sliding the scales along, one quickly sees the number of possible combinations. This figure shows the 4 possible combinations of a sum of 9.

Scales for other solids may also be constructed.

side of the cardboard are the same length. There should be just enough space between the circular pieces for the hexagons to turn on the string. Fasten the circles to the string with a drop of glue. Make an arrow on the circle next to the smaller hexagon. (See Figure C.)



C

To operate the hexawhirl, hold a loop in each hand and swing the cardboard pieces around and around (25 or more times) until the loops of string are twisted. Pull the loops until the twisting is undone, release to allow string to twist the other way, and pull again. With practice you can make the hexagons spin rapidly between the circles. Stop, and see which sides of the hexagons are in line with the arrow (2 on the smaller, 3 on the larger, for instance). Experiment to find out if the results are similar to those obtained by throwing two dice.